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On the Uniquely Solvability of Cauchy Problem and Dependences of Parameters for a Certain Class of Linear Functional Differential Equations

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

The objectives of this paper is to investigate the Cauchy problem of the type $(\mathcal{L}x)(t)\quad \dot{x}(t) - p(t)x(t) = f(t).\quad x(a) = \alpha, t\epsilon[a, b]$ and to establishes the necessary and sufficient conditions for it solvability if the nxn matrix ρ are sum able. Thus the above equations can be re written as $x(t) = X(t)\int_a^t X^{-1}(s)f(s)ds + X(t)\alpha$. Where X is a fundamental matrix such that $X(a)$ is the identity matrix, also can be represented as the general solution of the equation of the type $\mathcal{L}x = f$. So the studying equations can be written as a boundary value problem of linear functional differential equations of the form $\mathcal{L}x = f, IX = \alpha$. The Green Operators was used to established the conditions that guarantee uniquely solvable bounded value problem of the type defined above. The paper also considered the case where the boundary value problems continuous dependence of parameters, to established conditions that guarantee uniquely solvability of the equations of the form $\mathcal{L}_0x = f, \mathcal{L}_0x = \alpha$ and $\mathcal{L}_kx = f, \mathcal{L}_kx = \alpha$. With the establishment of these two arguments, the objectives of this paper was established.

Keywords: Uniquely; solvability; cauchy problem; dependence of parameters and functional equation.

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1 Introduction

The problem of Solvability of Cauchy problems of Linear Functional Differential Equations of Various order has been the subject of many investigations. Many Research papers and books have been devoted to the study of Cauchy problems of Functional Differential Equations of various degree. Among others see the literature: (1) gives the Introduction to the Theory of Functional Differential Equations. It is interesting to note that the boundary value problem plays the same role in the theory of Functional Differential Equations as the Cauchy problem does in the theory of ordinary Differential Equations (2) Was on the conditions for optimal and unique solvability of the Cauchy problem for the first order linear Functional Differential Equations. (3,4 5), considered and established conditions for solvability of Cauchy problems of higher order, and periodic boundary value problems for Functional Differential Equations. On the other hand (6), Considered conditions that guarantee the stable and the unstable State of a certain Class of Delay Differential Equations. (7) Considered Stability of periodic solutions of non- linear Delay Differential Equations and established the conditions that guarantee it globally asymptotically properties. (8) gives sufficient conditions that guarantee unique solvability of the Dirichlet problem of second – order Functional Differential Equations. (9,10) , was on two – dimensional systems, and gives the sufficient conditions for their solvability for Linear Functional Differential Equations.

In this paper, we study and established the sufficient and necessary conditions that guarantee the Unique Solvability of the Cauchy problem of the type algebraic form, with the matrix IX which is not equal to zero. Also the case when the boundary value problems are continuous dependences on parameters to established it Unique and bounded Solvability. My approach in this study improved on the literature as in the authors, to the case where more than two arguments of the studying equations were established, as in the case of one argument in the authors in (2,3).

2 Preliminaries

In this paper we investigate the Cauchy problem of the type

$$(\mathcal{L}x)(t) \quad \dot{x}(t) - p(t)x(t) = f(t). \quad x(a) = \alpha, t\epsilon[a, b] \tag{1.1}$$

Which is uniquely solvable for any $\alpha \in \mathbb{R}^n$ and sum able f, if the elements of the nxn matrix ρ are sum able. Thus, the representation of the solution of equations (1.1) is given as

$$x(t) = X(t) \int_a^t X^{-1}(s)f(s)ds + X(t)\alpha \tag{1.2}$$

Where X is a fundamental matrix such that $X(a)$ is the identity matrix, also (1.2) can be represented as the general solution of the equation of the type:

$$\mathcal{L}x = f. \qquad (1.2b).$$

Therefore the boundary value problem plays the same role in the theory of functional differentials equations as the Cauchy problem does in the theory of ordinary differential equations. Equation (1.1) can be re written as a boundary value problem of the form

$$\mathcal{L}x = f, IX = \alpha \tag{1.3}$$

We be considered without the assumption that the number m of boundary conditions equal $n.$ Denote $\rho = rank\ IX$. In the case $\rho > 0$, we may assume without loss of generality that the determinant of the rank ρ composed from the element in the left top of the matrix IX does not become zero. We now choose the fundamental vector as follows. In the case that $\rho > 0$, the elements $x_{1---,}\ x_\rho$ are selected in such a way that $1^i\ xj = \delta ij, i, j, ---, \rho$ (δij is the kronecker symbol). If $0 \le \rho < n$, the homogeneous problem $\mathcal{L}x =$

$0, IX = 0\ has\ n-\rho$ linearly independent solutions $u_{1----,}u_{n-\rho.}$ Everywhere below will be taken as a fundamental vector, the vector $X = (u_{n,----,}u_n)$ if $\rho = 0$, the vector $X = (x_{1,---,}x_{\rho,}\ u_{1,---}u_{n-\rho})$ if $o < \rho < n$, and the vector $X = (x_{1,---,}x_n)\ if\ \rho = n$.

The question about solvability of equation (1.3), is the question about solvability of a linear algebraic system with the matrix IX.

Let consider the case if $\rho = m < n$, the problem is solvable (but not uniquely solvable) for any $f \epsilon B, \alpha = \{x^1, ---, \alpha^m\} \epsilon \mathbb{R}^m$. To obtain the representation of the solution in this case, we can supplement the functional $I^{1,--,}I^m$ by additional functional $I^{M+i}, --I^n$ such that

$$\det(I^{m+i}, uj)i, j^{n-m} \neq 0 \tag{1.4}$$

The determinant of the problem

$$\mathcal{L}x = f,\ I^1x = \alpha^1, --, I^nx = \alpha^n \tag{1.5}$$

does not become zero, and therefore equation (1.3) is uniquely solvable. Using the Green operator G for equation (1.3), we can represent the solutions for equation (1.3) in the form of

$$X = Gf + \sum_{i=1}^{n} \alpha^i\, xi + \sum_{i=1}^{n-m} ci\ uj \tag{1.6}$$

Where $c_{1,---,}c_{n-m}$ are arbitrary constants. In all the other cases, equation (1.3) is not everywhere solvable. The conditions of it solvability can be obtained using the Green operators of any uniquely solvable bounded value problem for the equation of the form $\mathcal{L}x = f.$ If equation (1.3) has a unique solution for each $f \in B, \alpha \in \mathbb{R}^m$ if and only if m= n, and the det $IX \neq 0$. And this is said to be the determinant of equation (1.3), which be represented as follows:

$$\begin{pmatrix} \varphi & A \\ \Phi & \Psi \end{pmatrix} \begin{pmatrix} \delta x \\ rx \end{pmatrix} = \begin{pmatrix} f \\ \alpha \end{pmatrix} \tag{1.7}$$

The operator

$$\begin{pmatrix} \varphi^* & \Phi^* \\ A^* & \Psi^* \end{pmatrix} : B^*\, X\, (\mathbb{R}^n) * \longrightarrow\ B^*\, X\ (R^m) * \tag{1.8}$$

The ad joint to equation (1.7) taking into account the isomorphism between the spaces $B^*\ X\ \ (\mathbb{R}^n) *$, is given as follows:

$$\begin{pmatrix} \varphi^* & \Phi^* \\ \mathrm{A}^* & \Psi^* \end{pmatrix} \begin{pmatrix} \omega \\ \mathrm{Y} \end{pmatrix} = \quad \begin{pmatrix} \mathcal{g} \\ \eta \end{pmatrix} \tag{1.9}$$

Corollary 2.1: The equation (1.7) is solvable if and only if the right –hand side $\{f, \alpha\}$ is orthogonal to all the solutions $\{\omega, \gamma\}$ of the homogeneous ad joint equation.

$$\begin{aligned} \varphi^*\omega + \Phi^*\mathrm{Y} &= 0 \\ A^*\omega + \psi^* &= 0 \end{aligned} \tag{1.10}$$

The condition of being orthogonal has the form

$$\{\omega, f\} + \{\gamma, \alpha\} = 0 \tag{1.11}$$

Theorem 2.0: Let $\mathcal{L}: D \longrightarrow B$ be a Noether operator with independent $\mathcal{L} = n.$ Then the dimension of the kernel $\mathcal{L}\ \geq n$ and also dim ker $\mathcal{L} = n$ if (1.2a) is a solvable for each $f\ \in B.$

Theorem 2.1: The following assertions are equivalent.

(a) $\mathbb{R}(\mathcal{L}) = B$
(b) dim ker $\mathcal{L} = n.$
(c) There exist a vector function $I : D \dashrightarrow \mathbb{R}^n$ such that equations (1.3, and 1.7) are uniquely solvable for each $f \in B, \alpha\, \mathbb{R}^n$.

Theorem 2.2: A linear bounded operator $B : B \dashrightarrow D$ is a Green operator of a boundary value problem of (1.3) if and only if the following conditions are fulfilled.

(a) G is a Noether operator, ind G = - n
(b) Ker G = {0}

Theorem 2.3: Let n be odd , let the function $r\,(t,s)$ do not decrease with respect to the second argument, and let at least one of the inequalities

$$\int_a^b [r(t,b) - r(t,a)]\, dt \; < \frac{(n-1)!}{(b-a)^{n-1}}, \tag{1.12}$$

$$ess\ \sup[\, r(t,b) - r\,(t,a)] < \frac{n!}{(b-a)^n} \quad t \in (a,b) \tag{1.13}$$

Hold

$$\text{Then } (\mathcal{L}x)(t) \;\; \underline{\text{def}}\;\; X^{(n)}\,(t) + \int_a^t x(s) ds\, r(t,x) = f(t) \tag{1.14}$$

Possesses P – property.

Theorem 2.4: Let $G\ and\ G_1$ be Green operators of the problems

$$\mathcal{L}x = f, \quad IX = \alpha$$
$$\mathcal{L}x = f, \;\; lix = \alpha \tag{1.15}$$

Let further, X be the fundamental vector of $\mathcal{L}x = 0$.

$$\text{Then } G = G_1 - x(IX)^{-1}\, IG_1 \tag{1.16}$$

Theorem 2.5: Assume that a boundary value problem of (1.3) is uniquely solvable.

Let $P : B \longrightarrow B$ be a linear bounded operator with bounded inverse P^{-1} . The Green operator of this problem has the representation

$$G = w_1(P + H) \tag{1.17}$$

Where $H : B \longrightarrow B$ is compact operator iff the principal part $\mathbb{Q}\ of \mathcal{L}$ may be represented in the form $\mathbb{Q} = P^{-1} + V, where\ V\colon B \longrightarrow B$ is a compact operator.

3 Proof of the Theorems

Poof of Theorem 2.1

The equivalence of the assertions (a) and (b) was established in the prove of Theorem 2.0.

Let dim ker $\mathcal{L} = n\ and\ I = [\mathrm{I}^1, --, \mathrm{I}^n)$, let this system be bi orthogonal to the bases $x_1, --, x_n$ of the kernel of $\mathcal{L}$: $I^i xj = \delta ij, i, j = 1, --, n,$ where δij is the kronecker symbol.

Then the equation (1.3) with such an I has the unique solution

$$x = X(\alpha - IV) + V \tag{3.1}$$

Where $X = (x_{1,---,}\ x_n)$ and v is any solution of $\mathcal{L}x = f$. This is seen by taking into account that $IX = E$. Conversely, if (1.3) is uniquely solvable for each $f\ and\ \alpha$, one can take the solutions of the problems

$$,\mathcal{L}x = 0,\ IX = \alpha i,\ \alpha j \in \mathbb{R}^n, i = 1,--,n. \tag{3.2}$$

As the bases $x_{1,} --,\ x_n$ if the matrix $(\alpha_1,--,\ \alpha_n)$ is invertible . Thus the equivalence of the assertions (b) and (c) is proved. **This complete the proof**.

Proof of Theorem 2.2

Let $\{G, X\}: B\ x\ \mathbb{R}^n \longrightarrow D$ be one –to-one mapping if G is the Green operator of equation (1.3), let G be such that (a) and (b) are fulfilled. Then dim ker $G^* = n. If\ I^1,--,I^n$ constitute a basis of ker G^* and $I = [I^1,--,\ I^n]$, then $R(G) = \ker I$. G is the Green operator of equation (1.3),

$$\text{Where } \mathcal{L}x = G^{-1}(x - uIx) + VIx, \tag{3.3}$$

G^{-1} is the inverse to $G : B \longrightarrow ker\ell$; $U = (u_{1,--}\ u_n)$, $uj \in D$, is a vector such that

$IU = E; and\ V = (v_{1,--,}\ v_n), vi \in B$ is an arbitrary vector. **This complete the proof.**

Proof of Theorem 2.3

Let $\tau \in (a, b)$ be fixed. Consider the equation

$$(\mathcal{L}^\tau(x)(t)\ \underline{\underline{\text{def}}}\ X^{(n)}(t) + \int_a^t x(s)ds\, r^\tau(t,s) = f(t), t \in [a,\tau] \tag{3.4}$$

The operator $\mathcal{L}^\tau$ is defined on the space of the functions $X: [a,\tau] \longrightarrow \mathbb{R}$. The boundary value problem

$$(\mathcal{L}^\tau x)(t) = f(t), x^{(k)}(\tau) = 0, k = 0 --, n-1, t \in [a,\tau] \tag{3.5}$$

Is equivalent to the equation $X = A^\tau x + g$, where the operator $A^\tau: C[a,\tau] \longrightarrow C[a,\tau]$ is defined by

$$(A^\tau x)(t) = -\int_a^t \frac{(t-s)^{n-1}}{(n-1)!}\{\int_a^s x(\xi)d\xi\ r(s,\xi)\}ds \quad t\epsilon[a,\tau], g(t) = \int_a^t \frac{(t-s)}{(n-1)!}^{n-1} f(s)ds \tag{3.6}$$

Condition (1.12) guarantees the estimate $\rho(A^\tau) < 1$. This implies, by equation (3.5) is uniquely solvable for each $\tau \in (a,b)$, and, besides, the Green operator G^τ of the problem is ant isotonic. The same assertion holds for equation (1.13). Consequently ,any solution x of the equation $\mathcal{L}x = f$, is the extension on (τ, b) of a solution x^τ of the equation $\mathcal{L}^\tau x = f$. It follows, in particular that the conditions of Theorem 2.3 guarantee that the Wronskian of the fundamental system $x_{1,--,}\ x_n$ of the solutions of $\mathcal{L}x = 0$ has no zeros solutions. This implies that the solution y of the problem.

$$\mathcal{L}x = f, X^{(i)}(\tau) = o, i = 0,--,n-1 \tag{3.7}$$

Satisfies the inequality $\begin{bmatrix} x_1(\tau) -- & x_n(\tau) & 0 \\ x_1{}^{(n-1)}(\tau) -- & x_n{}^{(n-1)}(\tau) & 0 \\ x_1{}^{(n)}(\tau) -- & x_n{}^{(n)}(\tau) & y^{(n)}(\tau) \end{bmatrix}$ (3.8)

$$= -\int_a^t (G\tau f)(s)ds\, r(\tau,s) + f(\tau)$$

Thus, if $f(t) \geq 0$ on (a,b), any solution x of the equation $\mathcal{L}x = f$ satisfies the inequality $(Mx)(t) \geq f(t) \geq 0$. Thus the operator P is isotonic. **This complete the proof**.

Proof of Theorem 2.5

Let $G = W(\mathbb{Q} - F)^{-1}$, $\mathbb{Q} = P^{-1} + V. Define\ V_1 = V - F.$
Then $(\mathbb{Q} - F)^{-1} = (P^{-1} + V_1)^{-1} = (1 + PV_1)^{-1}P = (1 + H_1)^p = P + H$ (3.9)

Where H: B $\longrightarrow B$ and $H_1\colon B \longrightarrow B$ are compact operators.

Conversely, if $(\mathbb{Q} - F)^{-1}\ P + H$, then

$$\mathbb{Q} = F + (P + H)^{-1} = F + (I + P^{-1}H)^{-1}P^{-1} =$$
$$F + (I + V_1)P^{-1} = P^{-1} + V \quad (3.10)$$

Where $V : B \longrightarrow B\ and\ V_1\colon B \longrightarrow B$ are compact operators. **This complete the proof**.

4 Proof of the Main Results

Consider the boundary value problem of the form

$$\bar{\mathcal{L}}\, y = f,\ \ \bar{\ell}\, y = \alpha \quad (4.1)$$

In the space $\bar{D}$. Since dim ker $\bar{\mathcal{L}} = n + \mu$, equation (4.1) may be uniquely and everywhere solvable only if $\mu = m - n$. But if $\mu > m - n$, it is necessary to add to m boundary conditions some more $\mu + n - m = 0$ and equation (4.1) becomes

$$\bar{\mathcal{L}}\, y = f,\ \ \tilde{\imath}\, y = \alpha, \tilde{\imath_1}\, y = \alpha \quad (4.2)$$

If $m + n - \mu > 0$ are called extended boundary value problems. Hence $\tilde{\imath_1}\colon \tilde{D} \longrightarrow \mathbb{R}^{\mu+n-m}$ is a linear bounded vector functional.

Theorem 4.1 Let $m = n$, and let equation (1.3) be uniquely solvable, and let $\bar{D} = D \oplus M^{\mu}$. For any linear extensions $\bar{\mathcal{L}} : \tilde{D} \longrightarrow B, \tilde{\imath} :\ \tilde{D} \longrightarrow \mathbb{R}^n\ of\ \mathcal{L} : D \longrightarrow B, and\ \ell : D \longrightarrow \mathbb{R}^n$, there exists a vector functional $\tilde{\imath_1} :\ \tilde{D} \longrightarrow \mathbb{R}^n$ such that equation (4.2) is uniquely solvable.

Proof. For any linear extension $\tilde{\imath}$ of vector functional ℓ, we have $\tilde{I}X = IX$. Therefore

det $\tilde{I}X\ \neq 0$. Let choose $y_1, --, y\mu$ in such a way that $\tilde{\imath}yi =\ 0$, I $= 1, --, \mu$. Letting

$$yi = \bar{y}i - \textstyle\sum_{j=1}^{n} cjxj \quad (4.3)$$

For a fundamental system $x_{1,} --,\ x_n,\ \bar{y}_1, --, \bar{y}_\mu$ of the solutions of the equation $\bar{\mathcal{L}}y = 0$, will introduce constants $c_1, --, c_n$, equation (4.3) becomes

$$\sum_{j=1}^{n} cj\widetilde{\ell^{k}}\, xj = \widetilde{\ell^{k}}\bar{y}i, \quad k = 1,---,n \tag{4.4}$$

With a determinant that is not equal to zero. Consider a system of functional $\widetilde{\ell^{n+1}} : \widetilde{D} \longrightarrow \mathbb{R}^{\prime},\ i = 1,--,\mu,$ such that $\Delta = \det\,(\widetilde{\ell^{n+1}}\ yj)^{\mu}\ i,j = 1\ \neq 0$ (4.5)

Hence the determinant of equation (4.2) with $\widetilde{\ell_1} = \left[\widetilde{\ell^{n+1}},---,\widetilde{\ell^{n+\mu}}\right]$ is equal to Δ.
$detIX \neq 0$. **This complete the proof**.

Theorem 4.2 . Let $\widetilde{D} = D \oplus M^{\mu}$. If equation (1.3) has a uniquely solvable extended problem, then $\mu \geq m - \rho$.

Proof. Let $\mu < m - \rho\,.\, if\ \rho = n$, then $\mu < m - n$. Therefore only the case $\rho < n$ needs the proof . Let $\widetilde{\mathcal{L}}\ \ and\ \tilde{I}$ be any linear extension on the space $\widetilde{D}$ of $\mathcal{L}\ and\ \ell$, respectively. If $\mu = m - n$, then the determinant of equation (4.1), which is order m, is equal to zero because it has nonzero elements only at the columns corresponding to $x_1\,,--,x_\rho,\ y_1,--,y_\mu, if\ \ \rho\ > 0$ or $y_1,---,y_\mu.\ If\ \rho = 0$, the number of such columns is equal to $\rho +\ \mu\ < m.\ Let\ \mu > m - n$, then the determinant of equation (4.2) is equal to zero. Really, the cofactors of the minors of the $(\mu + n - m)th$ order composed from the elements of the rows corresponding to the vector functional $\tilde{I}_l$ are determinants of the m^{th} order. These determinants are equal to zero. This complete the proof.

5 Continuous Dependence on Parameters

Consider the sequences $\{\mathcal{L}_k\},\{\tilde{\ell}_k\}$ of bounded linear operators $\mathcal{L}_k\colon D_k - B_k,\ ind\,\mathcal{L}_k = n$, and bounded linear vector functionals $\mathcal{L}_k\colon D_k - |R^n$ with linearly independent components $k = 0,1,\ldots$, we will assume that $\mathcal{L}_k - \mathcal{L}_0$ and that $\mathcal{L}_k u_k - \mathcal{L}_0 u_0\ \ if\ u_k -\ u_0$.

Let consider the boundary value problem

$$\mathcal{L}_0 x = f, \mathcal{L}_0 x = \alpha \tag{5.1}$$

be uniquely solvable. Consider the question about conditions which provide the unique solvability of the problems

$$\mathcal{L}_k x = f, \mathcal{L}_k x = \alpha \tag{5.2}$$

for all k large enough and also the convergence $x_k \overset{\mathcal{P}}{\rightarrow} x_0$ for any sequences $\{f_k\}$ and $\{\alpha_k\}$, $f_k - f_0, \alpha_k - \alpha_0$. Here x_k is the solution of the problem

$$\mathcal{L}_k X = f_k, \mathcal{L}_k X = \alpha_x \tag{5.3}$$

And x_0 is the solution of the problem. $\mathcal{L}_0 x = f_0,\ \mathcal{L}_0 x = \alpha_0$ (5.4).

5.1 Preliminaries

Definition 5.1.1: A system $\mathcal{P} = (\mathcal{P}_k),\ k = 1, 2, \ldots$, of linear bounded operators $\mathcal{P}_k\colon E_0 \rightarrow E_k$ is said to be connecting for E_0 and $E_k,\ k = 1,2,\ldots, if\ \lim_{k\to\infty}\|\mathcal{P}_k u\|\ E_k = \|u\|E_0\ \ for\ any\ u \in E_0$.

Definition 5.1.2: The sequence $\{u_k\}$, $u_k \epsilon E_k$, is said to be $\mathcal{P}$ −convergent to $u \epsilon E_0$, which is denoted by $u_k \overset{\mathcal{P}}{\rightarrow} u_0$, if $\lim_{k\to\infty}\|u_k - \mathcal{P}_k u_0\| E_k = 0$

Definition 5.1.3: The sequence $\{u_k\}$, $u_k \epsilon E_k$, is said to be $\mathcal{P}$ −compact if any of its subsequences includes a $\mathcal{P}$ −convergent sebsequence.

Definition 5.1.4: The sequence $\{A_k\}$, is said to be $\mathcal{P}Q$ −Convergent to A_0, which is denoted by $A_k - A_0$, if the sequence $\{A_k u_k\}$ is Q −Convergent to $A_0 u_0$ for any sequence $\{u_k\}$, $u_k \in E_k$, that is, $\mathcal{P}$ −Convergent to $u_0 \epsilon E_0$.

5.2 Prove of the main results

The following Lemma we help us to prove the main result.

Lemma 5.1: $M_k u - \mathcal{L}_0 u$ for any $u \epsilon D_0$ if and only if and only if and only if $\mathcal{L}_k - \mathcal{L}_0$

Proof. Let $\mathcal{L}_k - \mathcal{L}_0$. Since $\mathcal{P}_k u - u$ and $sup_k\|\mathcal{H}_k^{-1}\| < \infty$, we have

$$M_k u - \mathcal{L}_0 u = \mathcal{H}_k^{-1}(\mathcal{L}_k \mathcal{P}_k u - \mathcal{H}_k \mathcal{L}_0 u) = 0 \quad (5.5)$$

Conversely, let $M_k u - \mathcal{L}_0 u$ for any $u \epsilon D_0$ and $u_k - u_0$. We have $\mathcal{L}_k u_k - \mathcal{H}_k \mathcal{L}_0 u = \mathcal{H}_k M_k \mathcal{P}_k^{-1} u_k - \mathcal{H}_k \mathcal{L}_0 u$ $= \mathcal{H}_k\{M_k(\mathcal{P}_k^{-1}u_k - u_0) + (M_k u_0 - \mathcal{L}_0 u_0)\} = 0$ (5.6)

Since $\mathcal{P}_k^{-1}u_k - u_0$, $M_k u_0 - \mathcal{L}_0 u_0, sup_k\|\mathcal{H}_k\| < \infty$, $sup_k\|M_k\| < \infty$.

This complete the proof.

Lemma 5.2: The operators Φ_k and F_k are continuously invertible (or not) simultaneously; $\Phi_k^{-1} - \Phi_0^{-1}$ if and only if $\mathrm{F}_k^{-1}y - \mathrm{F}_0^{-1}y$ for any $y \in B_0 \times \mathbb{R}^n$.

Where $\Phi_k = [\mathcal{L}_k, \ \mathcal{L}\mathcal{P}_k^{-1}] : D_k - B_k \times \mathbb{R}^n$

$$F_k = [\mathcal{H}_k^{-1}\mathcal{L}_k\mathcal{P}_k, \mathcal{L}] : D_0 - B_0 \times \mathbb{R}^n \quad (\Phi_0 = F_0) \quad (5.7)$$

Proof: Simultaneous inevitability follows from the representation $\Phi_k = \Phi_k F_k \mathcal{P}_k^{-1}$

Let $F_k^{-1}y - F_0^{-1}y$ for any $y \in B_0 \times \mathbb{R}^n$ and $y_k - y_0$, $y_k \epsilon B_k \times \mathbb{R}^n$. We have

$$\Phi_k^{-1}y_k - \mathcal{P}_k\Phi_0^{-1}y_0 = \mathcal{P}_k F_k^{-1}\Phi_k^{-1}(y_k - \mathcal{Q}_k y_0) + \mathcal{P}_k(F_k^{-1}y_0 - F_0^{-1}y_0) \quad (5.8)$$

From here, it follows that $\Phi_k^{-1} - \Phi_0^{-1}$. Conversely, let $\Phi_k^{-1} - \Phi_0^{-1}$. we have $F_k^{-1}y - F_0^{-1}y = \mathcal{P}_k^{-1}(\Phi_k^{-1}\Phi_k y - \mathcal{P}_k\Phi_0^{-1}y)$. Which imply, $F_k^{-1} - F_0^{-1}y$. **The end of the proof**

Theorem 5.1: Let equation (5.1) be uniquely solvable. Then equation (5.2) are uniquely solvable for all sufficiently large k; and for any sequences$\{f_k\}, \{\alpha_k\}$, $f_k - f_0, \alpha_k - \alpha_0$, the solutions x_k of equations (5.3) are $\mathcal{P} - Convergent$ to the solution x_0 of equation (5.4) if and only if there exists a vector functional $\lambda: D_0 - \mathbb{R}^n$ such that the equation

$$\mathcal{H}_k^{-1}\mathcal{L}_k\mathcal{P}_k x = f, \qquad 1x = \alpha \quad (5.9)$$

Are uniquely solvable for $k = 0$ and all sufficiently large k for any right-hand side $\{f, \alpha\} \epsilon B_0 \times \mathbb{R}^n$ and also the convergence $V_k \rightarrow V_0$ of the solution $V_k \epsilon D_0$ of equations (5.9) holds.

The Proof of the Main Results Theorem 5.1

Let us represent the operator

$$[\mathcal{L}_k, l_k] \text{ in the form } [\mathcal{L}_k, l_k] = [\mathcal{L}_k, l\mathcal{P}_k^{-1}] + [0, l_k - l\mathcal{P}_k^{-1}] \tag{5.10}$$

Since $\mathcal{L}_k - \mathcal{L}_0$ $and\ l\mathcal{P}_k^{-1} u_k - lu_0$ if $u_k - u_0$, we have

$$\Phi_k = [\mathcal{L}_k, l\mathcal{P}_k^{-1}] - [\mathcal{L}_0, l] = \Phi_0 \tag{5.11}$$

By virtue of lemma 5.2, there exist, for all sufficiently large k, continuous inverses

$$\Phi_k^{-1} = [\mathcal{L}_k, l\mathcal{P}_k^{-1}]^{-1} : B_k \times \mathbb{R}^n - D_k \tag{5.12}$$

And $\Phi_k{}^{-1}$ -- $\Phi_0{}^{-1}$. Thus, taking into account, if $A_k - A_0$, them $sup_k \ \| A_K \| < \ \infty$.

If a sequence $\{Y_K\}$ of the elements of a Banach space converges to y_0 by the norm, we will denote this fact henceforth by $y_k - \ y_0$. Condition (1) is fulfilled for the sequence $\{\Phi_k\}$.

Let consider the sequence of the operators $C_k = [0, l_k - l\mathcal{P}_k^{-1}]$:

$$D_k - B_k \times \mathbb{R}^n, k = 1,2 \ldots \tag{5.13}$$

Let $U_k - U_0$ then $l_k \ U_k - l_0 U_0$ due to the assumption (c) of the theorem and $|\mathcal{P}_k^{-1} \ U_k - \ U_0| = 0$ since $\mathcal{P}_k^{-1} \ U_k \ - \ U_0$. Therefore

$$C_k - C_0 = [\,0, l_0 - l\,]. \tag{5.14}$$

If the sequence $\{ U_k \}$, belong to D_k , is bounded, from the estimate

$$|\,(l_k - 1\mathcal{P}_k^{-1})\, U_k \,| \leq \| \ l_k - 1\mathcal{P}_k^{-1} \ \| \| \ U_k \ \| \ D_k \tag{5.15}$$

And the boundedness in common of the norms $\| \ l_k - 1\mathcal{P}_k^{-1} \ \|$, there follow boundedness in $\mathbb{R}^n$ and, consequently , compactness of the sequence $\{ \ l_k - 1\mathcal{P}_k^{-1} \}U_k$. So , the sequence $\{C_k \, U_k \}$ is Q – Compact. Let consider the case, when taken l_0 as the vector functional l. In other words, the operators

$$F_k = [\,\mathcal{H}_k{}^{-1} \ l_k \ B_{k,} l_0\,]: D_0 \to B_0 \times \mathbb{R}^n \tag{5.16}$$

Have, for all sufficiently large k , continuous inverses $F_k{}^{-1}$ and $F_k{}^{-1} \ y \ - \ F_0{}^{-1} \ y$ for any $y \in B_0 \times \mathbb{R}^n$. By virtue of lemma 5.2, it is sufficient to verify that for all sufficiently large k, the operators

$$Q_k = \ [\,{}_k,\ l_0 \ \mathcal{P}_k^{-1}\,] : D_k - B_k \times \mathbb{R}^n \tag{5.17}$$

Have continuous inverses with $Q_k^{-1} \ - \ Q_0^{-1}$. We have

$$Q_k = [\ {}_k,\ l_k] + [\,0,\ l_0 \ \mathcal{P}_k^{-1} - \ l_k\,] \tag{5.18}$$

Under the condition

$$B_k = [\ {}_k,\ l_k] \ - [\ {}_0,\ l_0\ ] = \ B_0 \tag{5.19}$$

For $k = 0$ and all sufficiently large k , there exist continuous inverse B_k^{-1} and also

$B_k^{-1} - B_0^{-1}$. Further we have

$$C_k = [0,\ l_0\ \mathcal{P}_k^{-1} - \ l_k\ ] - [0,0] = C_0 \tag{5.20}$$

Really, if $U_k \overset{\mathcal{P}}{\rightarrow} U_0$, then

$$[\ l_0\ \mathcal{P}_k^{-1} - \ l_k\ ]\, U_k = l_0\ \mathcal{P}_k^{-1}(U_k - \mathcal{P}_k\, U_0\,) - (\, l_k\ U_k - l_0 U_0\ ) = 0 \tag{5.21}$$

$Q_k = [\ \mathcal{L}_k,\ l_0\,\mathcal{P}_k^{-1}\ ] = B_k + C_k$ are Fredholm operators and ker $\Phi_0 = \{0\}$. Thus there exist, for all sufficiently large k, continuous inverse $\Phi_k{}^{-1}$ with $\Phi_k{}^{-1} - \Phi_0{}^{-1}$.

This complete the proof.

6 Conclusion

In this paper, we established the sufficient and necessary conditions that guarantee the Unique Solvability and continuous dependences of parameters of Cauchy problem for a certain class of Linear Functional Differential Equations. Obviously, the authors in [1,2,3,4,5,6,7,8,9 and 10] considered the Solvability of Cauchy problem of Linear Functional Differential Equations of various order. My approach and results of this paper improved on authors [2,3] to the case where more than two arguments of the studying equations were established. Hence , the results obtained in [1,2,3,4,5,6,7,8,9 and 10] are not the same in this paper, which implies that the results of this study are essentially new.

Competing Interests

Authors have declared that no competing interests exist.

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