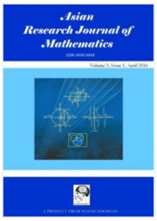

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Optimal Convex Combination Bounds for Toader Mean

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Authors' contributions

This work was carried out in collaboration between all authors. Aurhor HZX designed the study, and wrote the first draft of the manuscript. Aurhors SYL and FJ managed the analyses of the study. Aurhor FJ managed the literature searches. All authors read and approved the final manuscript.

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Abstract

In this paper, the authors prove that the double inequalities

$$\alpha_1 T(a,b) + (1-\alpha_1)\,H(a,b) < T\left[A(a,b),G(a,b)\right] < \beta_1 T(a,b) + (1-\beta_1)\,H(a,b),$$

$$\alpha_2 T(a,b) + (1-\alpha_2)\,G(a,b) < T\left[A(a,b),G(a,b)\right] < \beta_2 T(a,b) + (1-\beta_2)\,G(a,b)$$

hold for all $a,b>0$ with $a \neq b$ if and only if $\alpha_1 \leq 1/2$,$\beta_1 \geq 3/5$,$\alpha_2 \leq 1/3$ and $\beta_2 \geq 1/2$.Here $T(a,b)$,$T\left[A(a,b),G(a,b)\right]$,$H(a,b)$ and $G(a,b)$ are the Toader, Toader-type, harmonic and geometric means of a and b , respectively.

Keywords: Toader mean; toader-type mean; harmonic mean; geometric mean; the complete elliptic integral.

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1 Introduction

For $p \in \mathbb{R}$ and $a, b > 0$ with $a \neq b$, the pth power mean $M_p(a,b)$, harmonic mean $H(a,b)$, geometric mean $G(a,b)$, arithmetic mean $A(a,b)$, quadratic mean $Q(a,b)$, centroidal mean $E(a,b)$, contra-harmonic mean $C(a,b)$, and Toader mean $T(a,b)$[1] are defined respectively by

$$M_p(a,b) = \left[(a^p + b^p)/2\right]^{1/p} (p \neq 0), M_0(a,b) = \sqrt{ab},$$

$$H(a,b) = \frac{2ab}{a+b}, G(a,b) = \sqrt{ab}, \tag{1.1}$$

$$A(a,b) = \frac{a+b}{2}, Q(a,b) = \sqrt{\frac{a^2+b^2}{2}},$$

$$E(a,b) = \frac{2\left(a^2+ab+b^2\right)}{3(a+b)}, C(a,b) = \frac{a^2+b^2}{a+b}, \tag{1.2}$$

$$T(a,b) = \frac{2}{\pi}\int_0^{\pi/2} \sqrt{a^2\cos^2 t + b^2\sin^2 t}dt = \begin{cases} \frac{2a}{\pi}\varepsilon\left(\sqrt{1-(b/a)^2}\right), a > b, \\ \frac{2b}{\pi}\varepsilon\left(\sqrt{1-(a/b)^2}\right), a < b. \end{cases} \tag{1.3}$$

where $\varepsilon(r) = \int_0^{\pi/2}\left(1 - r^2\sin^2 t\right)^{1/2} dt, r \in (0,1)$ is the complete elliptic integral of the second kind. The pth power mean $M_p(a,b)$ is strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$, symmetric and homogeneous of degree 1. Then it is well known that the inequalities[2],[3]

$$\begin{aligned} H(a,b) = M_{-1}(a,b) < G(a,b) = M_0(a,b) < A(a,b) = M_1(a,b) \\ < T(a,b) < E(a,b) < Q(a,b) = M_2(a,b) < C(a,b) \end{aligned} \tag{1.4}$$

hold for all $a, b > 0$ with $a \neq b$.

The Toader mean $T(a,b)$ is well known in mathematical literature for many years, it satisfies

$$T(a,b) = R_E\left(a^2, b^2\right)$$

and

$$T(1,r) = \frac{2}{\pi}\varepsilon\left(\sqrt{1-r^2}\right)$$

for all $a, b > 0$ with $a \neq b$,where

$$R_E(a,b) = \frac{1}{\pi}\int_0^{+\infty} \frac{\left[a(t+b) + b(t+a)\right]t}{(t+a)^{3/2}(t+b)^{3/2}}dt$$

stands for the symmetric complete elliptic integral of the second kind (see [4],[5],[6]), therefore it cannot be expressed in terms of the elementary transcendental functions.

We need to note a simple fact: if $R_1(a,b)$,$R_2(a,b)$,$R(a,b)$ are means of distinct positive numbers a and b with $R_1(a,b) < R_2(a,b)$, then $R\left[R_1(a,b), R_2(a,b)\right]$ is also a mean and satisfies inequalities

$$R_1(a,b) < R\left[R_1(a,b), R_2(a,b)\right] < R_2(a,b).$$

Applying the fact, we can obtian

$$G(a,b) = M_0(a,b) < T\left[A(a,b), G(a,b)\right] < A(a,b) = M_1(a,b) \tag{1.5}$$

hold for all $a, b > 0$ with $a \neq b$.

Recently, the Toader mean has been the subject of intensive research. In particular, many remarkable inequalities for Toader mean and its generating can be found in the literature [7],[8],[9],[10],[11],[12],[13].

In [14], Vuorinen conjectured that

$$M_{3/2}(a,b) < T(a,b)$$

for all $a, b > 0$ with $a \neq b$. This conjecture was proved by Qiu and Shen [15], and Barnard et al. [16], respectively. Alzer and Qiu [17] presented the best possible upper power mean bound for the Toader mean as follows:

$$T(a,b) < M_{\log 2/\log(\pi/2)}(a,b)$$

for all $a, b > 0$ with $a \neq b$.

Neuman [4], Kazi and Neuman [5] proved that the inequalities

$$\frac{(a+b)\sqrt{ab}-ab}{AGM(a,b)} < T(a,b) < \frac{4(a+b)\sqrt{ab}+(a-b)^2}{8AGM(a,b)},$$

$$T(a,b) < \frac{1}{4}\left[\sqrt{\left(2+\sqrt{2}\right)a^2+\left(2-\sqrt{2}\right)b^2}+\sqrt{\left(2+\sqrt{2}\right)b^2+\left(2-\sqrt{2}\right)a^2}\right],$$

hold for all $a, b > 0$ with $a \neq b$, where $AGM(a,b)$ is the arithmetic-geometric mean of a and b.

In [18],[2],[19],[20],[21] the authors proved that the double inequalities

$$\begin{gathered}
\alpha_1 Q(a,b) + (1-\alpha_1) A(a,b) < T(a,b) < \beta_1 Q(a,b) + (1-\beta_1) A(a,b), \\
\alpha_2 E(a,b) + (1-\alpha_2) A(a,b) < T(a,b) < \beta_2 E(a,b) + (1-\beta_2) A(a,b), \\
\alpha_3 C(a,b) + (1-\alpha_3) A(a,b) < T(a,b) < \beta_3 C(a,b) + (1-\beta_3) A(a,b), \\
\alpha_4 C(a,b) + (1-\alpha_4) H(a,b) < T(a,b) < \beta_4 C(a,b) + (1-\beta_4) H(a,b), \\
\alpha_5 [C(a,b) - H(a,b)] + A(a,b) < T(a,b) < \beta_5 [C(a,b) - H(a,b)] + A(a,b), \\
\alpha_6 Q(a,b) + (1-\alpha_6) H(a,b) < T(a,b) < \beta_6 Q(a,b) + (1-\beta_6) H(a,b)
\end{gathered}$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq 1/2$, $\beta_1 \geq (4-\pi)/\left[\left(\sqrt{2}-1\right)\pi\right]$, $\alpha_2 \leq 3/4$, $\beta_2 \geq 12/\pi - 3$, $\alpha_3 \leq 1/4$, $\beta_3 \geq 4/\pi - 1, \alpha_4 \leq 5/8, \beta_4 \geq 2/\pi, \alpha_5 \leq 1/8, \beta_5 \geq 2/\pi - 1/2, \alpha_6 \leq 5/6$, $\beta_6 \geq 2\sqrt{2}/\pi$.

From inequalities (1.4) and (1.5) we clearly see that

$$H(a,b) < G(a,b) < T[A(a,b), G(a,b)] < A(a,b) < T(a,b) \tag{1.6}$$

hold for all $a, b > 0$ with $a \neq b$.

The main purpose of this paper is to present the best possible parameters $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that the double inequalities

$$\alpha_1 T(a,b) + (1-\alpha_1) H(a,b) < T[A(a,b), G(a,b)] < \beta_1 T(a,b) + (1-\beta_1) H(a,b),$$

$$\alpha_2 T(a,b) + (1-\alpha_2) G(a,b) < T[A(a,b), G(a,b)] < \beta_2 T(a,b) + (1-\beta_2) G(a,b)$$

hold for all $a, b > 0$ with $a \neq b$.

2 Basic Knowledge and Lemmas

In order to prove our main results we need some basic knowledge and Lemma, which we present in this section.

For $r \in (0,1)$,the complete elliptic integrals of the first and second kinds are defined by [22]

$$\kappa(r) = \int_0^{\pi/2} \left(1 - r^2 \sin^2 t\right)^{-1/2} dt$$

and

$$\varepsilon(r) = \int_0^{\pi/2} \left(1 - r^2 \sin^2 t\right)^{1/2} dt.$$

respectively. We clearly see that

$$\kappa\left(0^+\right) = \varepsilon\left(0^+\right) = \pi/2, \kappa\left(1^-\right) = +\infty, \varepsilon\left(1^-\right) = 1,$$

here $\kappa(r)$ and $\varepsilon(r)$ satisfy the formulas (see[21], Appendix E, p. 474-475)

$$\frac{d\kappa(r)}{dr} = \frac{\varepsilon(r) - \left(1 - r^2\right)\kappa(r)}{r\left(1 - r^2\right)}, \quad \frac{d\varepsilon(r)}{dr} = \frac{\varepsilon(r) - \kappa(r)}{r},$$
$$\frac{d\left[\kappa(r) - \varepsilon(r)\right]}{dr} = \frac{r\varepsilon(r)}{1 - r^2}, \quad \varepsilon\left(\frac{2\sqrt{r}}{1 + r}\right) = \frac{2\varepsilon(r) - \left(1 - r^2\right)\kappa(r)}{1 + r}.$$

Lemma 2.1. *(1) [22, Theorem 3.21(1)] The function* $r \mapsto \left[\varepsilon(r) - \left(1 - r^2\right)\kappa(r)\right]/r^2$ *is strictly increasing from* $(0,1)$ *onto* $(\pi/4, 1)$.

(2) [22, Exercise 3.43(11)] The function $r \mapsto \left[\kappa(r) - \varepsilon(r)\right]/r^2$ *is strictly increasing from* $(0,1)$ *onto* $(\pi/4, +\infty)$.

Lemma 2.2. *(1) The function* $r \mapsto \sqrt{1 - r^2}\left[\kappa(r) - \varepsilon(r)\right]/r^2$ *is strictly decreasing from* $(0,1)$ *onto* $(0, \pi/4)$.

(2) The function $r \mapsto \sqrt{1 - r^2}\left[\varepsilon(r) - \left(1 - r^2\right)\kappa(r)\right]/r^2$ *is strictly decreasing from* $(0,1)$ *onto* $(0, \pi/4)$.

Proof. For part (1), Let

$$\varphi_1(r) = \frac{\sqrt{1 - r^2}\left[\kappa(r) - \varepsilon(r)\right]}{r^2}. \tag{2.1}$$

Then simple computations lead to

$$\varphi_1\left(0^+\right) = \pi/4, \varphi_1\left(1^-\right) = 0, \tag{2.2}$$

$$\varphi_1'(r) = \frac{\phi_1(r)}{r^3\sqrt{1 - r^2}}. \tag{2.3}$$

where

$$\phi_1(r) = 2\varepsilon(r) - \left(2 - r^2\right)\kappa(r),$$
$$\phi_1\left(0^+\right) = 0, \tag{2.4}$$

$$\phi_1'(r) = -\frac{r^4}{1 - r^2}\left[\frac{\varepsilon(r) - \left(1 - r^2\right)\kappa(r)}{r^2}\right]. \tag{2.5}$$

It follows from Lemma 2.1(1) and (2.4)-(2.5) lead to

$$\phi_1'(r) < 0, \tag{2.6}$$

for all $r \in (0,1)$. Hence $\phi_1(r)$ is strictly increasing on $(0,1)$ directly from (2.6).

Therefore, part (1) follows from (2.2) and (2.3) together with the monotonicity of $\phi_1(r)$.

For part (2), Let

$$\varphi_2(r) = \frac{\sqrt{1-r^2}\left[\varepsilon(r) - \left(1-r^2\right)\kappa(r)\right]}{r^2}. \tag{2.7}$$

Then simple computations lead to

$$\varphi_2\left(0^+\right) = \pi/4, \varphi_2\left(1^-\right) = 0, \tag{2.8}$$

$$\varphi_2'(r) = \frac{1}{r\sqrt{1-r^2}}\left[\varepsilon(r) - 2\frac{\varepsilon(r) - \left(1-r^2\right)\kappa(r)}{r^2}\right]. \tag{2.9}$$

From (2.9) and Lemma 2.1(1) together with the monotonicity of $\varepsilon(r)$ we get

$$\varphi_2'(r) < \frac{1}{r\sqrt{1-r^2}}\left[\frac{\pi}{2} - 2\times\frac{\pi}{4}\right] = 0 \tag{2.10}$$

for $r \in (0,1)$.

Therefore, part (2) follows easily from (2.8) and (2.10).

Lemma 2.3. *Let $p \in (0,1)$,$r \in (0,1)$ and*

$$f(r) = p\frac{\varepsilon(r) - \left(1-r^2\right)\kappa(r)}{r^2} + \frac{\kappa(r) - \varepsilon(r)}{r^2} - \pi(1-p). \tag{2.11}$$

Then the following statements are true:

(1) $p = 3/5$, then $f(r) > 0$ for all $r \in (0,1)$.

(2) $p = 1/2$, then there exists $r_1 \in (0,1)$ such that $f(r) < 0$ for $r \in (0,r_1)$ and $f(r) > 0$ for $r \in (r_1,1)$.

Proof. For part (1), if $p = 3/5$, then (2.11) becomes

$$f(r) = \frac{3}{5}\frac{\varepsilon(r) - \left(1-r^2\right)\kappa(r)}{r^2} + \frac{\kappa(r) - \varepsilon(r)}{r^2} - \frac{2}{5}\pi. \tag{2.12}$$

It follows from Lemma 2.1(1)-(2) and (2.12) that

$$f(r) > \frac{3}{5}\times\frac{\pi}{4} + \frac{\pi}{4} - \frac{2}{5}\pi = 0$$

for all $r \in (0,1)$.

For part (2),if $p = 1/2$, then Lemma 2.1(1)-(2) and (2.12) lead to

$$f\left(0^+\right) = -\frac{\pi}{8}, f\left(1^-\right) = +\infty \tag{2.13}$$

and $f(r)$ is strictly increasing on $(0,1)$.

Therefore, part (2) follows from (2.13) and the monotonicity of $f(r)$.

Lemma 2.4. *Let $p \in (0,1)$,$r \in (0,1)$ and*

$$g(r) = p\frac{\sqrt{1-r^2}\left[\varepsilon(r) - \left(1-r^2\right)\kappa(r)\right]}{r^2} + \frac{\sqrt{1-r^2}\left[\kappa(r) - \varepsilon(r)\right]}{r^2} - \frac{1}{2}\pi(1-p). \tag{2.14}$$

Then the following statements are true:

(1) $p = 1/3$, *then* $g(r) < 0$ *for all* $r \in (0, 1)$.

(2) $p = 1/2$, *then there exists* $r_2 \in (0, 1)$ *such that* $g(r) > 0$ *for* $r \in (0, r_2)$ *and* $g(r) < 0$ *for* $r \in (r_2, 1)$.

Proof. For part (1), if $p = 1/3$, then (2.14) becomes

$$g(r) = \frac{1}{3}\frac{\sqrt{1-r^2}\left[\varepsilon(r) - \left(1-r^2\right)\kappa(r)\right]}{r^2} + \frac{\sqrt{1-r^2}\left[\kappa(r) - \varepsilon(r)\right]}{r^2} - \frac{\pi}{3}. \tag{2.15}$$

It follows from Lemma 2.2(1)-(2) and (2.15) that

$$g(r) < \frac{1}{3} \times \frac{\pi}{4} + \frac{\pi}{4} - \frac{\pi}{3} = 0$$

for all $r \in (0, 1)$.

For part (2),if $p = 1/2$, then Lemma 2.2(1)-(2) and (2.14) lead to

$$g\left(0^+\right) = \frac{\pi}{8}, g\left(1^-\right) = -\frac{\pi}{4} \tag{2.16}$$

and $g(r)$ is strictly decreasing on $(0, 1)$.

Therefore, part (2) follows from (2.16) and the monotonicity of $g(r)$.

3 Main Results

Theorem 3.1. *The double inequality*

$$\alpha_1 T(a,b) + (1-\alpha_1) H(a,b) < T[A(a,b), G(a,b)] < \beta_1 T(a,b) + (1-\beta_1) H(a,b)$$

holds for all $a, b > 0$ *with* $a \neq b$ *if and only if* $\alpha_1 \leq 1/2$ *and* $\beta_1 \geq 3/5$.

Proof. Since $H(a,b)$,$G(a,b)$,$A(a,b)$ and $T(a,b)$ are symmetric and homogenous of degree 1 and $G(a,b) < T[A(a,b), G(a,b)] < A(a,b)$. Without loss of generality, we assume that $a > b > 0$. Let $r = (a-b)/(a+b) \in (0, 1)$ and $p \in (0, 1)$.Then from (1.1) and (1.3) leads to

$$T(a,b) = \frac{2}{\pi} A(a,b) \left[2\varepsilon(r) - \left(1-r^2\right)\kappa(r)\right], \tag{3.1}$$

$$H(a,b) = A(a,b)\left(1-r^2\right), T[A(a,b), G(a,b)] = \frac{2}{\pi} A(a,b)\,\varepsilon(r). \tag{3.2}$$

It follows from (3.1)-(3.2) lead to

$$\frac{T[A(a,b), G(a,b)] - H(a,b)}{T(a,b) - H(a,b)} = \frac{\frac{2}{\pi}\varepsilon(r) - \left(1-r^2\right)}{\frac{2}{\pi}\left[2\varepsilon(r) - (1-r^2)\kappa(r)\right] - (1-r^2)}, \tag{3.3}$$

$$pT(a,b) + (1-p) H(a,b) - T[A(a,b), G(a,b)] = A(a,b) F(r), \tag{3.4}$$

where

$$F(r) = \frac{2}{\pi} p \left[2\varepsilon(r) - \left(1-r^2\right)\kappa(r)\right] + (1-p)\left(1-r^2\right) - \frac{2}{\pi}\varepsilon(r),$$

$$F\left(0^+\right) = 0, \tag{3.5}$$

$$F\left(1^-\right) = \frac{2}{\pi}(2p-1), \tag{3.6}$$

$$F'(r) = \frac{2}{\pi} r \left[p\frac{\varepsilon(r) - \left(1-r^2\right)\kappa(r)}{r^2} + \frac{\kappa(r) - \varepsilon(r)}{r^2} - \pi(1-p)\right] = \frac{2}{\pi} r f(r) \tag{3.7}$$

where $f(r)$ is defined by (2.11).

We divide the proof into two cases.

Case 1 $p = 1/2$. Then (3.6) becomes

$$F\left(1^-\right) = 0, \tag{3.8}$$

It follows from Lemma 2.4(2) and (3.7) that there exists $r_1 \in (0,1)$ such that $F(r)$ is strictly decreasing on $(0, r_1]$ and strictly increasing on $[r_1, 1)$. Therefore,

$$T[A(a,b), G(a,b)] > \frac{1}{2}T(a,b) + \frac{1}{2}H(a,b) \tag{3.9}$$

follows from (3.4)-(3.5) and (3.8) together with the piecewise monotonicity of $F(r)$.

Case 2 $p = 3/5$.Then Lemma 2.4(1) and (3.7) lead to the conclusion that $F(r)$ is strictly increasing on $(0,1)$. Therefore,

$$T[A(a,b), G(a,b)] < \frac{3}{5}T(a,b) + \frac{2}{5}H(a,b) \tag{3.10}$$

follows from (3.4)-(3.5) and the monotonicity of $F(r)$.

Note that

$$\lim_{r\to 0^+} \frac{\frac{2}{\pi}\varepsilon(r) - \left(1-r^2\right)}{\frac{2}{\pi}\left[2\varepsilon(r) - (1-r^2)\kappa(r)\right] - (1-r^2)} = \frac{3}{5}, \tag{3.11}$$

$$\lim_{r\to 1^-} \frac{\frac{2}{\pi}\varepsilon(r) - \left(1-r^2\right)}{\frac{2}{\pi}\left[2\varepsilon(r) - (1-r^2)\kappa(r)\right] - (1-r^2)} = \frac{1}{2}. \tag{3.12}$$

Therefore, Theorem 3.1 follows from (3.9) and (3.10) together with the following statements.

- If $p > 1/2$,then (3.3) and (3.12) imply that there exists $0 < \delta_1 < 1$, such that

  $$T[A(a,b), G(a,b)] < pT(a,b) + (1-p)H(a,b)$$

  for all $a > b > 0$ with $(a-b)/(a+b) \in (0, \delta_1)$.
- If $p < 3/5$,then (3.3) and(3.11) imply that there exists $0 < \delta_2 < 1$, such that

  $$T[A(a,b), G(a,b)] > pT(a,b) + (1-p)H(a,b)$$

  for all $a > b > 0$ with $(a-b)/(a+b) \in (1-\delta_2, 1)$.

Theorem 3.2. *The double inequality*

$$\alpha_2 T(a,b) + (1-\alpha_2)G(a,b) < T[A(a,b), G(a,b)] < \beta_2 T(a,b) + (1-\beta_2)G(a,b)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_2 \leq 1/3$ and $\beta_2 \geq 1/2$.

Proof. Without loss of generality, we assume that $a > b > 0$. Let $r = (a-b)/(a+b) \in (0,1)$.Then from (3.1), (3.2) and $G(a,b) = A(a,b)\sqrt{1-r^2}$ we get

$$\frac{T[A(a,b), G(a,b)] - G(a,b)}{T(a,b) - G(a,b)} = \frac{\frac{2}{\pi}\varepsilon(r) - \sqrt{1-r^2}}{\frac{2}{\pi}\left[2\varepsilon(r) - (1-r^2)\kappa(r)\right] - \sqrt{1-r^2}}, \tag{3.13}$$

$$pT(a,b) + (1-p)G(a,b) - T[A(a,b), G(a,b)] = A(a,b)G(r) \tag{3.14}$$

where

$$G(r) = p\frac{2}{\pi}\left[2\varepsilon(r) - \left(1-r^2\right)\kappa(r)\right] + (1-p)\sqrt{1-r^2} - \frac{2}{\pi}\varepsilon(r),$$

$$G\left(0^+\right) = 0, \tag{3.15}$$

$$G\left(1^-\right) = \frac{2}{\pi}(2p-1), \tag{3.16}$$

$$G'(r) = \frac{2r}{\pi\sqrt{1-r^2}}\left[p\frac{\sqrt{1-r^2}\left[\varepsilon(r)-\left(1-r^2\right)\kappa(r)\right]}{r^2} + \frac{\sqrt{1-r^2}\left[\kappa(r)-\varepsilon(r)\right]}{r^2} - \frac{1}{2}\pi(1-p)\right]$$

$$= \frac{2r}{\pi\sqrt{1-r^2}}g(r) \tag{3.17}$$

where $g(r)$ is defined by (2.14).

We divide the proof into two cases.

Case 1 $p = 1/3$. Then Lemma 2.5(1) and (3.17) lead to the conclusion that $G(r)$ is strictly decreasing on $(0, 1)$. Therefore,

$$T[A(a,b), G(a,b)] > \frac{1}{3}T(a,b) + \frac{2}{3}G(a,b), \tag{3.18}$$

follows from (3.14)-(3.15) and the monotonicity of $G(r)$.

Case 2 $p = 1/2$.Then (3.16) becomes

$$G\left(1^-\right) = 0, \tag{3.19}$$

It follows from Lemma 2.5(2) and (3.17) that there exists $r_2 \in (0, 1)$ such that $G(r)$ is strictly increasing on $(0, r_2]$ and strictly decreasing on $[r_2, 1)$. Therefore,

$$T[A(a,b), G(a,b)] < \frac{1}{2}T(a,b) + \frac{1}{2}G(a,b) \tag{3.20}$$

follows from (3.14)-(3.15) and (3.19) together with the piecewise monotonicity of $G(r)$.

As an application, Corollary 3.3 follows immediately from Theorems 3.1-3.2 and Lemma 2.1(1). We establish new inequalities for the complete elliptic integral of the second kind.

Corollary 3.3. *Let $I_A(x)$ be the characteristic function and defined by*

$$I_A \equiv I_A(x) = \begin{cases} 1 & , x \in A \\ 0 & , x \notin A. \end{cases}$$

Let

$$L_1(r) = \frac{\pi}{4}\left(2-r^2\right), L_2(r) = \frac{\pi}{8}\left(4\sqrt{1-r^2}+r^2\right),$$

$$U_1(r) = \frac{1}{2}\left[3r^2 + \pi\left(1-r^2\right)\right], U_2(r) = \frac{1}{2}\left[\pi\sqrt{1-r^2} + 2r^2\right]$$

The double inequality

$$L_1(r) \cdot I_{\{r>\frac{2}{3}\sqrt{2}\}} + L_2(r) \cdot I_{\{r\le\frac{2}{3}\sqrt{2}\}} < \varepsilon(r) < U_1(r) \cdot I_{\left\{r\le\frac{\sqrt{\pi^2-2\pi}}{\pi-1}\right\}} + U_2(r) \cdot I_{\left\{r>\frac{\sqrt{\pi^2-2\pi}}{\pi-1}\right\}} \tag{3.21}$$

holds for all $r \in (0, 1)$.

Remark 3.1. Recently, the complete elliptic integrals have attracted the attention of many reseachers. In [23], Barnard et al. established that

$$\varepsilon(r) \le \frac{\pi}{2}\left(\frac{2-r^2}{2}\right)^{\frac{1}{2}} \tag{3.22}$$

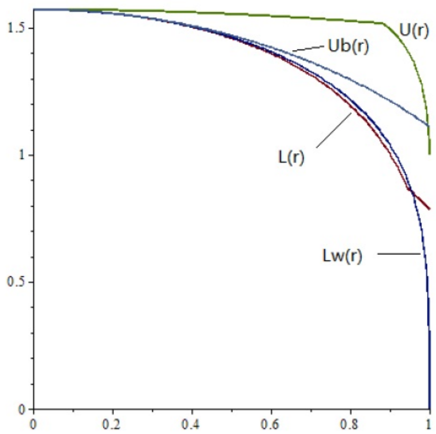

Fig 1. Comparisons of L(r) with Lw(r),and U(r) with Ub(r).

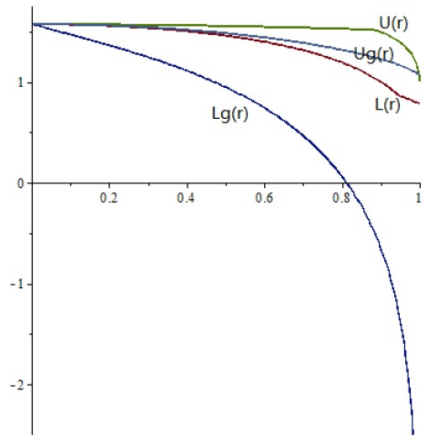

Fig 2. Comparisons of L(r) with Lg(r),and U(r) with Ug(r).

In [24], Wang et al. established that

$$\varepsilon(r) > \frac{\pi}{2}\left(1-r^2\right)^{\frac{1}{4}} \tag{3.23}$$

for all $r \in (0,1)$.

Guo and Qi [25] proved that

$$\frac{\pi}{2} - \frac{1}{2}\ln\left[\frac{(1+r)^{1-r}}{(1-r)^{1+r}}\right] < \varepsilon(r) < \frac{\pi-1}{2} + \frac{1-r^2}{4r}\ln\left(\frac{1+r}{1-r}\right) \tag{3.24}$$

for all $r \in (0,1)$.

Let

$$L(r) = L_1(r)\cdot I_{\{r>\frac{2}{3}\sqrt{2}\}} + L_2(r)\cdot I_{\{r\le\frac{2}{3}\sqrt{2}\}},$$

$$U(r) = U_1(r)\cdot I_{\{r\le\frac{\sqrt{\pi^2-2\pi}}{\pi-1}\}} + U_2(r)\cdot I_{\{r>\frac{\sqrt{\pi^2-2\pi}}{\pi-1}\}},$$

$$Lw(r) = \frac{\pi}{2}\left(1-r^2\right)^{\frac{1}{4}},$$

$$Ub(r) = \frac{\pi}{2}\left(\frac{2-r^2}{2}\right)^{\frac{1}{2}},$$

$$Lg(r) = \frac{\pi}{2} - \frac{1}{2}\ln\left[\frac{(1+r)^{1-r}}{(1-r)^{1+r}}\right],$$

$$Ug(r) = \frac{\pi-1}{2} + \frac{1-r^2}{4r}\ln\left(\frac{1+r}{1-r}\right).$$

Fig. 1 and Fig. 2 show that the bounds in (3.21) for $\varepsilon(r)$ are better than that in (3.22)-(3.24) for some $r \in (0,1)$, respectively.

4 Conclusion

We study Optimal Convex Combination Bounds for Toader Mean in terms of harmonic mean and geometric mean. We establish new inequalities for the complete elliptic integral of the second kind.Further research in this field can be carried out.

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Competing Interests

Authors have declared that no competing interests exist.

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