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Generalized a:k:m-Fibonacci Numbers

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Original Research Article

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Abstract

The a:k:m-Fibonacci sequence is defined recursively by $F_{a:k:m,n}\begin{cases} f_{a:k:m,n+2} = kf_{a:k:m,n+1} + amf_{a:k:m,n}, a,k,m,n \geq 1 \\ f_{a:k:m,1} = 1, f_{a:k:m,2} = k \end{cases}$. We introduce the generalized a:k:m-Fibonacci sequence $H_{a:k:m,n}\{h_{a:k:m,n+2} = kh_{a:k:m,n+1} + amh_{a:k:m,n}, n \geq 1,$ with arbitrary integers $h_{a:k:m,1}$ and $h_{a:k:m,2}$, and study some important properties relating to the Pascal type triangle generated from this sequence. The results are extended to negative values of k and m, an important concept which brings on board Lehmer type sequences. Most importantly, generalized a:k:m-Fibonacci sequences provide a means to unify ideas that are otherwise treated independently. The theory of a:k:m-Fibonacci numbers is therefore a unification theory.

Keywords: a:k:m-Fibonacci numbers; a:k:m-Lucas numbers; Jacobsthal numbers; k-Fibonacci numbers; k-Lucas numbers; k-Jacobsthal numbers; k-Pell numbers; Lehmer type sequences.

1 Introduction

Number Theory must be understood as a branch of geometry, this perhaps was passively argued by Euclid when he treated this important subject alongside "traditional" geometry in his famous work *The Elements* [1].

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Many ancient practising geometers, most of whom are known as philosophers elsewhere, including most notably Pythagoras, allowed number to take first place in their studies and to greatly influence their systems of belief and by extension their lives. This may sound a bit ridiculous but Pythagoras is quoted as saying in one place, *"All is number"*. Livio [2] notes, *"Pythagoras (ca. 572 – 497 BC) may have been the first person who was both an influential natural philosopher and a charismatic spiritual philosopher – a scientist and a religious thinker. In fact, he is credited with introducing the words "philosophy", meaning love of wisdom, and "mathematics" – the learned disciplines."*

Many numeric systems have been discovered and/or invented since times ancient. In modern mathematics various numeric systems are being studied, including the Lucas numbers, Pell numbers, Jacobsthal numbers, etc. see [3 – 26]. There is, however, one numeric system which seems to claim an unfair media coverage. This system is the so-called Fibonacci sequence, the historical background of which is perhaps discussed in an unusual context by Singh [24] stating, *"What are generally referred to as the Fibonacci numbers and the method for their formation were given by Virahanka (between A.D. 600 and 800), Gopala (prior to A.D. 1135), and Hemacandra (c. A.D. 1150), all prior to L. Fibonacci (c. A.D. 1202)."* The Fibonacci numbers are

$$F_n = 1,1,2,3,5,8,\ldots \tag{1.1}$$

generated by the recurrence relation

$$f_{n+2} = f_{n+1} + f_n, f_1 = f_2 = 1 \;\; n \geq 1 \tag{1.2}$$

The k-Fibonacci numbers introduced by Falcon and Plaza [9] and seriously studied by other researchers [10-23] are defined by

$$F_{k,n}\begin{cases} f_{k,n+2} = kf_{k,n+1} + f_{k,n}, \;\; n,k \geq 1 \\ f_{k,1} = 1, f_{k,2} = k \end{cases} \tag{1.3}$$

In solving quadratic equations, four types of sequences arise:

Type I

The a:k:m-Fibonacci numbers defined by

$$F_{a:k:m,n}\begin{cases} f_{a:k:m,n+2} = kf_{a:k:m,n+1} + amf_{a:k:m,n}, a,k,m,n \geq 1 \\ f_{a:k:m,1} = 1, f_{a:k:m,2} = k \end{cases} \tag{1.4}$$

are introduced in [19] based on the solutions of the quadratic equation

$$ax^2 - kx - m = 0 \tag{1.5}$$

The two parameters that we obtain are:

$$\sigma_a^{k:m} = \frac{k+\sqrt{k^2+4am}}{2} \tag{1.6}$$

and

$$\tau_a^{k:m} = \frac{k-\sqrt{k^2+4am}}{2} \tag{1.7}$$

k-Fibonacci numbers arise when $a = m = 1$ and k-Jacobsthal numbers arise when, in the notation of equation (1.4), $k = 1, am \geq 1$. Again, in the notation of equation (1.4), k-Pell sequence is obtained when $k = 2, am \geq 1$.

In this manuscript, we study the basic properties and the Pascal type triangle of the generalized a:k:m-Fibonacci sequence defined by

$$H_{a:k:m,n}\{h_{a:k:m,n+2} = kh_{a:k:m,n+1} + amh_{a:k:m,n}, n \geq 1 \quad (1.8)$$

with arbitrary integers $h_{a:k:m,1}$ and $h_{a:k:m,2}$. For interest's sake the reader may extend the results to the following types of sequences:

Type II

$$H_{a:-k:m,n}\{h_{a:-k:m,n+2} = -kh_{a:-k:m,n+1} + amh_{a:-k:m,n}, n \geq 1 \quad (1.9)$$

Parameters: $\sigma_a^{-k:m} = \frac{-k+\sqrt{k^2+4am}}{2}$, $\tau_a^{-k:m} = \frac{-k-\sqrt{k^2+4am}}{2}$

Type III

$$H_{a:k:-m,n}\{h_{a:k:-m,n+2} = kh_{a:k:-m,n+1} - amh_{a:k:-m,n}, n \geq 1 \quad (1.10)$$

Parameters: $\sigma_a^{k:-m} = \frac{k+\sqrt{k^2-4am}}{2}$, $\tau_a^{k:-m} = \frac{k-\sqrt{k^2-4am}}{2}$

Here, Lehmer type sequences are studied e.g.

$$F_{2:1:-1,n} = 1,1,-1,-3,-1,5,7,-3, \ldots \quad (1.11)$$

Type IV

$$H_{a:-k:-m,n}\{h_{a:-k:-m,n+2} = -kh_{a:-k:-m,n+1} - amh_{a:-k:-m,n}, n \geq 1 \quad (1.12)$$

Parameters: $\sigma_a^{-k:-m} = \frac{-k+\sqrt{k^2-4am}}{2}$, $\tau_a^{-k:m} = \frac{-k-\sqrt{k^2-4am}}{2}$

Remark 1.1

The parameters $\sigma_a^{\pm k:\pm m}$ and $\tau_a^{\pm k:\pm m}$ are obtained from multiplying the solutions of the corresponding quadratic equation by the variable a. One can work directly with the solutions as demonstrated in [19] but the given parameters are handier.

2 Basic Properties of $H_{a:k:m,n}$

In [19] these two identities are proved:

$$a^{-1}\sigma_a^{k:m} - a^{-1}f_{a:k:m,n}\tau_a^{k:m} = mf_{a:k:m,n-1}, n \geq 1 \quad (2.1)$$

$$f_{a:k:m,n+1} = f_{a:k:m,n}\sigma_a^{k:m} + (\tau_a^{k:m})^n, n \geq 1 \quad (2.2)$$

Lemma 2.1: Binet's formula for $F_{a:k:m,n}$

$$f_{a:k:m,n} = \frac{(\sigma_a^{k:m})^n - (\tau_a^{k:m})^n}{\sigma_a^{k:m} - \tau_a^{k:m}}, n \geq 1 \quad (2.3)$$

Proof

Notice that

$$f_{a:k:m,n} = \frac{f_{a:k:m,n+1} - am f_{a:k:m,n-1}}{k} \tag{2.4}$$

From equations (2.1) and (2.2), equation (2.4) becomes

$$f_{a:k:m,n} = \frac{\sigma_a^{k:m} f_{a:k:m,n} + \left(\tau_a^{k:m}\right)^n - \left(\sigma_a^{k:m}\right)^n + \sigma_a^{k:m} f_{a:k:m,n}}{k}$$

It follows that

$$f_{a:k:m,n}\left(k - 2\sigma_a^{k:m}\right) = \left(\tau_a^{k:m}\right)^n - \left(\sigma_a^{k:m}\right)^n$$

$$\therefore f_{a:k:m,n} = \frac{\left(\tau_a^{k:m}\right)^n - \left(\sigma_a^{k:m}\right)^n}{k - 2\sigma_a^{k:m}} = \frac{\left(\sigma_a^{k:m}\right)^n - \left(\tau_a^{k:m}\right)^n}{\sigma_a^{k:m} - \tau_a^{k:m}}$$

Theorem 2.1: Binet's formula for $H_{a:k:m,n}$

$$h_{a:k:m,n} = \frac{\alpha\left(\sigma_a^{k:m}\right)^n - \beta\left(\tau_a^{k:m}\right)^n}{\sigma_a^{k:m} - \tau_a^{k:m}}, n \geq 1 \tag{2.5}$$

where

$$\alpha = \frac{h_{a:k:m,2} - h_{a:k:m,1}\tau_a^{k:m}}{\sigma_a^{k:m}} \tag{2.6}$$

$$\beta = \frac{h_{a:k:m,2} - h_{a:k:m,1}\sigma_a^{k:m}}{\tau_a^{k:m}} \tag{2.7}$$

Derivation

Having proved equation (2.3), assume there exist two constants α, β such that equation (2.5) holds. It follows that

$$h_{a:k:m,1} = \frac{\alpha\left(\sigma_a^{k:m}\right)^1 - \beta\left(\tau_a^{k:m}\right)^1}{\sigma_a^{k:m} - \tau_a^{k:m}}, h_{a:k:m,2} = \frac{\alpha\left(\sigma_a^{k:m}\right)^2 - \beta\left(\tau_a^{k:m}\right)^2}{\sigma_a^{k:m} - \tau_a^{k:m}}$$

Solving for α, β equations (2.6) and (2.7) are obtained.

Equation (2.5) becomes a powerful tool in proving Theorems 2.2 through 2.5.

Theorem 2.2: d'Ocagne's Identity for $H_{a:k:m,n}$

$$h_{a:k:m,r}h_{a:k:m,n+1} - h_{a:k:m,n}h_{a:k:m,r+1} = (-am)^n \alpha\beta f_{a:k:m,r-n}, n \geq 1 \tag{2.8}$$

Proof

$$h_{a:k:m,r}h_{a:k:m,n+1} - h_{a:k:m,n}h_{a:k:m,r+1} =$$

$$\left(\frac{\alpha\left(\sigma_a^{k:m}\right)^r - \beta\left(\tau_a^{k:m}\right)^r}{\sigma_a^{k:m} - \tau_a^{k:m}}\right)\left(\frac{\alpha\left(\sigma_a^{k:m}\right)^{n+1} - \beta\left(\tau_a^{k:m}\right)^{n+1}}{\sigma_a^{k:m} - \tau_a^{k:m}}\right) - \left(\frac{\alpha\left(\sigma_a^{k:m}\right)^n - \beta\left(\tau_a^{k:m}\right)^n}{\sigma_a^{k:m} - \tau_a^{k:m}}\right)\left(\frac{\alpha\left(\sigma_a^{k:m}\right)^{r+1} - \beta\left(\tau_a^{k:m}\right)^{r+1}}{\sigma_a^{k:m} - \tau_a^{k:m}}\right)$$

$$= \frac{\alpha\beta\big((\sigma_a^{k:m})^r(\tau_a^{k:m})^{n+1}(-1+\sigma_a^{k:m}(\tau_a^{k:m})^{-1}) - (\sigma_a^{k:m})^{n+1}(\tau_a^{k:m})^r(1-(\sigma_a^{k:m})^{-1}\tau_a^{k:m})\big)}{(\sigma_a^{k:m}-\tau_a^{k:m})^2}$$

$$= \frac{\alpha\beta\big((\sigma_a^{k:m})^r(\tau_a^{k:m})^n(\sigma_a^{k:m}-\tau_a^{k:m}) - (\sigma_a^{k:m})^n(\tau_a^{k:m})^r(\sigma_a^{k:m}-\tau_a^{k:m})\big)}{(\sigma_a^{k:m}-\tau_a^{k:m})^2}$$

$$= (\sigma_a^{k:m}\tau_a^{k:m})^n\alpha\beta\frac{(\sigma_a^{k:m})^{r-n}-(\tau_a^{k:m})^{r-n}}{\sigma_a^{k:m}-\tau_a^{k:m}}$$

$$= (-am)^n\alpha\beta f_{a:k:m,r-n}$$

Theorem 2.3: Catalan's Identity for $H_{a:k:m,n}$

$$h_{a:k:m,n}^2 - h_{a:k:m,n+r}h_{a:k:m,n-r} = (-am)^{n-r}\alpha\beta f_{a:k:m,r}^2, n \geq 1 \tag{2.9}$$

Proof

$$h_{a:k:m,n}^2 - h_{a:k:m,n+r}h_{a:k:m,n-r} =$$

$$\left(\frac{\alpha(\sigma_a^{k:m})^n-\beta(\tau_a^{k:m})^n}{\sigma_a^{k:m}-\tau_a^{k:m}}\right)^2 - \left(\frac{\alpha(\sigma_a^{k:m})^{n+r}-\beta(\tau_a^{k:m})^{n+r}}{\sigma_a^{k:m}-\tau_a^{k:m}}\right)\left(\frac{\alpha(\sigma_a^{k:m})^{n-r}-\beta(\tau_a^{k:m})^{n-r}}{\sigma_a^{k:m}-\tau_a^{k:m}}\right)$$

$$= \frac{\alpha\beta((\sigma_a^{k:m})^{n+r}(\tau_a^{k:m})^{n-r} + (\sigma_a^{k:m})^{n-r}(\tau_a^{k:m})^{n+r} - 2(\sigma_a^{k:m})^n(\tau_a^{k:m})^n)}{(\sigma_a^{k:m}-\tau_a^{k:m})^2}$$

$$= (\sigma_a^{k:m}\tau_a^{k:m})^{n-r}\alpha\beta\frac{((\sigma_a^{k:m})^{2r} - 2(\sigma_a^{k:m})^r(\tau_a^{k:m})^r + (\tau_a^{k:m})^{2r}}{(\sigma_a^{k:m}-\tau_a^{k:m})^2}$$

$$= (-am)^{n-r}\alpha\beta f_{a:k:m,r}^2$$

The new identities proved in theorems 2.4 and 2.5 transform $H_{a:k:m,n}$ to $F_{a:k:m,n}$. This means the self-referential character of $F_{a:k:m,n}$ under the transformation.

Theorem 2.4

$$\frac{amh_{a:k:m,n}^2 + h_{a:k:m,n+1}^2 - h_{a:k:m,0}(h_{a:k:m,2n+2} + amh_{a:k:m,2n})}{\alpha\beta} = f_{a:k:m,2n+1}, n \geq 1 \tag{2.10}$$

Proof

$$\frac{amh_{a:k:m,n}^2 + h_{a:k:m,n+1}^2 - h_{a:k:m,0}(h_{a:k:m,2n+2} + amh_{a:k:m,2n})}{\alpha\beta} =$$

$$\frac{1}{\alpha\beta}\Bigg[am\left(\frac{\alpha(\sigma_a^{k:m})^n-\beta(\tau_a^{k:m})^n}{\sigma_a^{k:m}-\tau_a^{k:m}}\right)^2 + \left(\frac{\alpha(\sigma_a^{k:m})^{n+1}-\beta(\tau_a^{k:m})^{n+1}}{\sigma_a^{k:m}-\tau_a^{k:m}}\right)^2$$
$$- \left(\frac{\alpha(\sigma_a^{k:m})^0-\beta(\tau_a^{k:m})^0}{\sigma_a^{k:m}-\tau_a^{k:m}}\right)\left(\left(\frac{\alpha(\sigma_a^{k:m})^{2n+2}-\beta(\tau_a^{k:m})^{2n+2}}{\sigma_a^{k:m}-\tau_a^{k:m}}\right)\right.$$
$$\left.+ am\left(\frac{\alpha(\sigma_a^{k:m})^{2n}-\beta(\tau_a^{k:m})^{2n}}{\sigma_a^{k:m}-\tau_a^{k:m}}\right)\right)\Bigg]$$

$$= \frac{-2am(\sigma_a^{k:m})^n(\tau_a^{k:m})^n - 2(\sigma_a^{k:m})^{n+1}(\tau_a^{k:m})^{n+1} + (\sigma_a^{k:m})^{2n+2} + (\tau_a^{k:m})^{2n+2} + am(\sigma_a^{k:m})^{2n} + am(\tau_a^{k:m})^{2n}}{(\sigma_a^{k:m} - \tau_a^{k:m})^2}$$

$$= \frac{-2(\sigma_a^{k:m})^n(\tau_a^{k:m})^n(am + \sigma_a^{k:m}\tau_a^{k:m}) + (\sigma_a^{k:m})^{2n+1}(\sigma_a^{k:m} + am(\sigma_a^{k:m})^{-1}) + (\tau_a^{k:m})^{2n+1}(\tau_a^{k:m} + am(\tau_a^{k:m})^{-1})}{\left(\sigma_a^{k:m} - \tau_a^{k:m}\right)^2}$$

$$= \frac{(\sigma_a^{k:m})^{2n+1}(\sigma_a^{k:m} - \tau_a^{k:m}) - (\tau_a^{k:m})^{2n+1}(\sigma_a^{k:m} - \tau_a^{k:m})}{(\sigma_a^{k:m} - \tau_a^{k:m})^2}$$

$$= \frac{(\sigma_a^{k:m})^{2n+1} - (\tau_a^{k:m})^{2n+1}}{\sigma_a^{k:m} - \tau_a^{k:m}}$$
$$= f_{a:k:m,2n+1}$$

Theorem 2.5

$$\frac{2amh_{a:k:m,n}h_{a:k:m,n+1} + kh^2_{a:k:m,n+1} - h_{a:k:m,0}(h_{a:k:m,2n+3} + amh_{a:k:m,2n+1}}{\alpha\beta} = f_{a:k:m,2n+2}, n \geq 1 \tag{2.11}$$

Proof

$$\frac{2amh_{a:k:m,n}h_{a:k:m,n+1} + kh^2_{a:k:m,n+1} - h_{a:k:m,0}(h_{a:k:m,2n+3} + amh_{a:k:m,2n+1}}{\alpha\beta} =$$
$$\frac{1}{\alpha\beta}\Bigg[2am\left(\frac{\alpha(\sigma_a^{k:m})^n - \beta(\tau_a^{k:m})^n}{\sigma_a^{k:m} - \tau_a^{k:m}}\right)\left(\frac{\alpha(\sigma_a^{k:m})^{n+1} - \beta(\tau_a^{k:m})^{n+1}}{\sigma_a^{k:m} - \tau_a^{k:m}}\right) + k\left(\frac{\alpha(\sigma_a^{k:m})^{n+1} - \beta(\tau_a^{k:m})^{n+1}}{\sigma_a^{k:m} - \tau_a^{k:m}}\right)^2$$
$$- \left(\frac{\alpha(\sigma_a^{k:m})^0 - \beta(\tau_a^{k:m})^0}{\sigma_a^{k:m} - \tau_a^{k:m}}\right)\left(\left(\frac{\alpha(\sigma_a^{k:m})^{2n+3} - \beta(\tau_a^{k:m})^{2n+3}}{\sigma_a^{k:m} - \tau_a^{k:m}}\right)\right.$$
$$\left.+ am\left(\frac{\alpha(\sigma_a^{k:m})^{2n+1} - \beta(\tau_a^{k:m})^{2n+1}}{\sigma_a^{k:m} - \tau_a^{k:m}}\right)\right)\Bigg]$$

$$= \frac{-2kam(\sigma_a^{k:m})^n(\tau_a^{k:m})^n - 2k(\sigma_a^{k:m})^{n+1}(\tau_a^{k:m})^{n+1} + (\sigma_a^{k:m})^{2n+3} + (\tau_a^{k:m})^{2n+3} + am(\sigma_a^{k:m})^{2n+1} + am(\tau_a^{k:m})^{2n+1}}{\left(\sigma_a^{k:m} - \tau_a^{k:m}\right)^2}$$

$$= \frac{-2k(\sigma_a^{k:m})^n(\tau_a^{k:m})^n(am + \sigma_a^{k:m}\tau_a^{k:m}) + (\sigma_a^{k:m})^{2n+2}(\sigma_a^{k:m} + am(\sigma_a^{k:m})^{-1}) + (\tau_a^{k:m})^{2n+2}(\tau_a^{k:m} + am(\tau_a^{k:m})^{-1})}{\left(\sigma_a^{k:m} - \tau_a^{k:m}\right)^2}$$

$$= \frac{(\sigma_a^{k:m})^{2n+2}(\sigma_a^{k:m} - \tau_a^{k:m}) - (\tau_a^{k:m})^{2n+2}(\sigma_a^{k:m} - \tau_a^{k:m})}{\left(\sigma_a^{k:m} - \tau_a^{k:m}\right)^2}$$

$$= \frac{(\sigma_a^{k:m})^{2n+2} - (\tau_a^{k:m})^{2n+2}}{\sigma_a^{k:m} - \tau_a^{k:m}}$$
$$= f_{a:k:m,2n+2}$$

3 Pascal Type Triangles for $H_{a:k:m,n}$

Pascal's numerical triangle has been widely studied and has influenced a lot of research and related work, see e.g [27-32]. Belbachir and Szalay [30] present an important literature review on the Pascal triangle. In this section, we state the construction rules and analyse the $H_{a:k:m,n}$ −Pascal triangle. We are interested in the summations of row and diagonal entries. Wells [31] gives proofs for the row and diagonal entries for the Pascal triangle.

3.1 Construction

Row 1: h_1
Row $n \geq 2$:
n elements are in row n. Let these elements be denoted $p_1, p_2, p_3, \dots p_n$. Let elements in row $n-1$ be denoted $q_1, q_2, q_3, \dots q_{n-1}$.
$p_1 = k^{n-2}h_2, \quad p_n = (am)^{n-1}h_1$
Let $2 \leq j \leq n-1$.
$p_j = amq_{j-1} + kq_j$

Table 3.1 gives the first four rows of the triangle so constructed.

Table 3.1 $H_{a:k:m,n}$ −Pascal Triangle

			h_1			
		h_2		amh_1		
	kh_2		$amh_2 + kamh_1$		$(am)^2h_1$	
k^2h_2		$2kamh_2 + k^2amh_1$		$2k(am)^2h_1 + (am)^2h_2$		$(am)^2h_2$

3.2 Analysis

Theorem 3.1

Let t_n *be the sum of entries in the* n^{th} *row of the* $H_{a:k:m,n}$ *−Pascal Triangle.*

$$t_n = (amh_1 + h_2)(k + am)^{n-2}, n \geq 2 \tag{3.1}$$

Proof

From the construction, it is deduced that

$$t_n = h_1 \sum_{r=0}^{n-2} \binom{n-2}{r} k^{n-r-2}(am)^{r+1} + h_2 \sum_{r=0}^{n-2} \binom{n-2}{r} k^{n-r-2}(am)^{r}, n \geq 2 \tag{3.2}$$

We provide proof by induction.

Base case: $n = 2$,

$$h_1 \binom{0}{0} am + h_2 \binom{0}{0} = h_1 am + h_2$$

which is true.

Inductive Hypothesis: Since formula holds for $n = 1$, *assume it holds for arbitrary* $n = i \geq 1$, *that is,*

$$h_1 \sum_{r=0}^{i-2} \binom{i-2}{r} k^{i-r-2}(am)^{r+1} + h_2 \sum_{r=0}^{i-2} \binom{i-2}{r} k^{i-r-2}(am)^{r} = (amh_1 + h_2)(k + am)^{i-2}, i \geq 2 \tag{3.3}$$

Inductive Conclusion: Formula must be shown to hold for $n = i + 1$, *that is,*

$$h_1 \sum_{r=0}^{i-1} \binom{i-1}{r} k^{i-r-1}(am)^{r+1} + h_2 \sum_{r=0}^{i-1} \binom{i-1}{r} k^{i-r-1}(am)^{r} = (amh_1 + h_2)(k + am)^{i-1}, i \geq 2 \tag{3.4}$$

We have that

$$(amh_1 + h_2)(k + am)^{i-1} = (amh_1 + h_2)(k + am)^{i-2}(k + am)$$

$$= h_1 \sum_{r=0}^{i-2} \binom{i-2}{r} k^{i-r-1}(am)^{r+1} + h_2 \sum_{r=0}^{i-2} \binom{i-2}{r} k^{i-r-1}(am)^{r} + h_1 \sum_{r=0}^{i-2} \binom{i-2}{r} k^{i-r-2}(am)^{r+2}$$
$$+ h_2 \sum_{r=0}^{i-2} \binom{i-2}{r} k^{i-r-2}(am)^{r+1}$$

$$= h_1 \sum_{r=0}^{i-1} \binom{i-1}{r} k^{i-r-1}(am)^{r+1} + h_2 \sum_{r=0}^{i-1} \binom{i-1}{r} k^{i-r-1}(am)^{r}$$

Proof is completed.

Theorem 3.2

Let d_n *be the sum of entries in the* n^{th} *rising diagonal of the* $H_{a:k:m,n}$ *−Pascal Triangle.*

$$d_n = h_n, n \geq 1 \tag{3.5}$$

Proof

For $n = 1, 2$ *it is straightforward that* $d_n = h_n$.

From the construction, it is deduced that

$$d_n = h_1 \sum_{r=0}^{\frac{n-3}{2}} \binom{n-r-3}{r} k^{n-2r-3}(am)^{r+1} + h_2 \sum_{r=0}^{\frac{n-3}{2}} \binom{n-r-2}{r} k^{n-2r-2}(am)^{r}, odd\ n \geq 3 \tag{3.6}$$

$$d_n = h_1 \sum_{r=0}^{\frac{n-4}{2}} \binom{n-r-3}{r} k^{n-2r-3}(am)^{r+1} + h_2 \sum_{r=0}^{\frac{n-2}{2}} \binom{n-r-2}{r} k^{n-2r-2}(am)^{r}, even\ n \geq 4 \tag{3.7}$$

Scenario I: odd n

We proceed by induction.

Base case: $n = 3$,

$$d_3 = h_1 \binom{0}{0} am + h_2 \binom{1}{0} k = amh_1 + kh_2 = h_3$$

Inductive Hypothesis: Since formula holds for $n = 1$, *assume it holds for arbitrary* $n = i \geq 1$, *that is,*

$$h_i = h_1 \sum_{r=0}^{\frac{i-3}{2}} \binom{i-r-3}{r} k^{i-2r-3}(am)^{r+1} + h_2 \sum_{r=0}^{\frac{i-3}{2}} \binom{i-r-2}{r} k^{i-2r-2}(am)^{r}, odd\ i \geq 3,$$

$$h_{i+1} = h_1 \sum_{r=0}^{\frac{i-3}{2}} \binom{i-r-2}{r} k^{i-2r-2}(am)^{r+1} + h_2 \sum_{r=0}^{\frac{i-1}{2}} \binom{i-r-1}{r} k^{i-2r-1}(am)^{r}, odd\ i \geq 3$$

Inductive Conclusion: Formula must be shown to hold for $n = i + 1$, *that is,*

$$h_{i+2} = h_1 \sum_{r=0}^{\frac{i-1}{2}} \binom{i-r-1}{r} k^{i-2r-1}(am)^{r+1} + h_2 \sum_{r=0}^{\frac{i-1}{2}} \binom{i-r}{r} k^{i-2r}(am)^r, odd\ i \geq 3$$

Notice that $h_{i+2} = kh_{i+1} + amh_i$

$$= h_1 \sum_{r=0}^{\frac{i-3}{2}} \binom{i-r-2}{r} k^{i-2r-1}(am)^{r+1} + h_2 \sum_{r=0}^{\frac{i-1}{2}} \binom{i-r-1}{r} k^{i-2r}(am)^r$$
$$+ h_1 \sum_{r=0}^{\frac{i-3}{2}} \binom{i-r-3}{r} k^{i-2r-3}(am)^{r+2} + h_2 \sum_{r=0}^{\frac{i-3}{2}} \binom{i-r-2}{r} k^{i-2r-2}(am)^{r+1}$$

$$= h_1 \sum_{r=0}^{\frac{i-1}{2}} \binom{i-r-1}{r} k^{i-2r-1}(am)^{r+1} + h_2 \sum_{r=0}^{\frac{i-1}{2}} \binom{i-r}{r} k^{i-2r}(am)^r$$

Formula is true.

Scenario II: even n

We proceed by induction.

Base case: $n = 4$,

$$d_4 = h_1 \binom{1}{0} kam + h_2 \binom{2}{0} k^2 + h_2 \binom{1}{1} am = kamh_1 + k^2h_2 + amh_2 = kh_3 + amh_2 = h_4$$

Inductive Hypothesis: Since formula holds for $n = 1$, *assume it holds for arbitrary* $n = i \geq 1$, *that is,*

$$h_i = h_1 \sum_{r=0}^{\frac{i-4}{2}} \binom{i-r-3}{r} k^{i-2r-3}(am)^{r+1} + h_2 \sum_{r=0}^{\frac{i-2}{2}} \binom{i-r-2}{r} k^{i-2r-2}(am)^r, even\ i \geq 4,$$

$$h_{i+1} = h_1 \sum_{r=0}^{\frac{i-2}{2}} \binom{i-r-2}{r} k^{i-2r-2}(am)^{r+1} + h_2 \sum_{r=0}^{\frac{i-2}{2}} \binom{i-r-1}{r} k^{i-2r-1}(am)^r, even\ i \geq 4$$

Inductive Conclusion: Formula must be shown to hold for $n = i + 1$, *that is,*

$$h_{i+2} = h_1 \sum_{r=0}^{\frac{i-2}{2}} \binom{i-r-1}{r} k^{i-2r-1}(am)^{r+1} + h_2 \sum_{r=0}^{\frac{i}{2}} \binom{i-r}{r} k^{i-2r}(am)^r, even\ i \geq 4$$

We have that $h_{i+2} = kh_{i+1} + amh_i$

$$= h_1 \sum_{r=0}^{\frac{i-2}{2}} \binom{i-r-2}{r} k^{i-2r-1}(am)^{r+1} + h_2 \sum_{r=0}^{\frac{i-2}{2}} \binom{i-r-1}{r} k^{i-2r}(am)^r$$
$$+ h_1 \sum_{r=0}^{\frac{i-4}{2}} \binom{i-r-3}{r} k^{i-2r-3}(am)^{r+2} + h_2 \sum_{r=0}^{\frac{i-2}{2}} \binom{i-r-2}{r} k^{i-2r-2}(am)^{r+1}$$

$$= h_1 \sum_{r=0}^{\frac{i-2}{2}} \binom{i-r-1}{r} k^{i-2r-1}(am)^{r+1} + h_2 \sum_{r=0}^{\frac{i}{2}} \binom{i-r}{r} k^{i-2r}(am)^r$$

Proof is completed.

Corollary 3.1

Let the sum of entries in n^{th} falling diagonal be denoted d_n'. The formula $d_{n+2}' = amd_{n+1}' + kd_n', n \geq 2$, holds.

4 Conclusion

It is interesting and especially important that the concept of this manuscript provides a means to treat the widely and independently studied sequences e.g. k-Fibonacci, k-Jacobsthal, and k-Pell, in a unified theoretical framework. For instance, Theorem 2.2 gives d'Ocagne's identity valid for k-Fibonacci, k-Jacobsthal, k-Pell, etc. sequences. The theory of a:k:m-Fibonacci sequences is, therefore, a unification theory. We particularly find Theorems 2.4 and 2.5 on the transformation of $H_{a:k:m,n}$ to $F_{a:k:m,n}$ demonstrating the self-referential character of the latter under the transformation, and the construction and analysis of the Pascal type triangles in Section 3, adding new knowledge to the field in the long term.

Competing Interests

Author has declared that no competing interests exist.

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