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Algorithm 1 and Algorithm 2 for Determining the Number pi (π)

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Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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ABSTRACT

Archimedes used the perimeter of inscribed and circumscribed regular polygons to obtain lower and upper bounds of π. Starting with two regular hexagons he doubled their sides from 6 to 12, 24, 48, and 96. Using the perimeters of 96 side regular polygons, Archimedes showed that $3+10/71<\pi<3+1/7$ and his method can be realized as a recurrence formula called the Borchardt-Pfaff-Schwab algorithm. Heinrich Dörrie modified this algorithm to produce better approximations to π than these based on Archimedes' scheme. Lower bounds generated by his modified algorithm are the same as from the method discovered earlier by cardinal Nicolaus Cusanus (XV century), and again re-discovered two hundred years later by Willebrord Snell (XVII century). Knowledge of Taylor series of the functions used in these methods allows to develop new algorithms. Realizing Richardson's extrapolation, it is possible to increase the accuracy of the constructed methods by eliminating some terms in their series. Two new methods are presented. An approximation of squaring the circle with high accuracy is proposed.

Keywords: Quadrature; circle; pi number; Archimedes; approximation; algorithm.

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1. INTRODUCTION

The first known rigours mathematical calculation of π was done by Archimedes. Archimedes' book "On the Measurements of a Circle", [1], written in the 3rd century B.C., contains three propositions. Proposition 3 represents the numerical computing of the number π. Archimedes used an algorithmic scheme based on doubling the number of sides in inscribed and circumscribed regular polygons. He started with the regular hexagons (*N* = 6) and doubled the number of their sides until *N* = 96. Archimedes obtained a series of two approximations, lower and upper, for length of the circumference of the circle with diameter equals to one (*d* = 1), thus consequently to the number π. Archimedes was able to determine the following bounds for the number π: $3+\frac{10}{71}<\pi<3+\frac{1}{7}$.

It was often suggested to combine these values to improve the approximation by taking their arithmetic average. This is correct, but it is possible to realize a better combination (see Table 1) than an arithmetical mean of these two bounds. Archimedes' estimations can be improved using only information already generated by the constructed polygons. Here two such improvements are proposed and presented. The proposed algorithms use only values obtained from the traditional and well-known methods. New created algorithms produce faster convergence to π than original techniques. Such approach already was realized for some other numerical methods. Table 1 shows the results for the regular polygons (*N* = 6, 12, 24, 48, and 96) and their combinations proposed in XVII century by Snell and later proved by Huygens [2]. Archimedes' approach is a true algorithm to obtain the value of π. The method is capable of generating an arbitrarily precise number π. The process is relatively slow in its convergence. It is also difficult to use this algorithm in direct calculations for a large number of sides. It is a similar situation as with Turing's machine and a modern computer. Theoretically all computable problems can be realized on both types of machines. It's only a difference and matter of time. There were many attempts to improve Archimedes' method. One such approach resulted in Pfaff-Borchardt-Schwab's (PBS) method developed in the XIX century. It was realized without using trigonometric functions.

The PBS method is defined by the following formulas: $a'=\frac{2ab}{a+b}, b'=\sqrt{a'b},$ new values a′, b′ are determined by the old values a, b - the values from the previous step. It's an iterative process and is easy to realize on a computer. Starting with a = 2√3 and b = 3; the values for circumscribed and inscribed regular 6-gons, we can generate the sequence of the intervals [b, a], b < a. The intervals contain π. It's π for the circle of the diameter one (d = 1), or for a unit circle (r = 1), and in this case it's half of its perimeter, which is also π.

Table 1 represents the approximations of the number π obtained by Archimedes' method (values a and b, from the inscribed and circumscribed polygons, respectively), their arithmetic average $(c=\frac{(a+b)}{2})$, and by using Snell's approach $(d=\frac{(2a+b)}{3})$, Here N determines the number of sides in regular polygons and $x=180^o/N$ is the central angle in the circle.

Ludolph van Ceulen (1540-1610) was the last person who performed great Archimedean calculations. He used 2^{62}-gons and obtained 39 places with 35 correct digits. The number is still called Ludolph's number in some parts of Europe. For example, in Poland it is called in Polish "liczba Ludolfina". Archimedes' method may be interpreted as a rectification problem. Its goal is to find the length of the arc of the considered circle. In this case, the method estimates the circumference of the circle (i.e. full arc for the angle 2π). A very simple and beautiful rectification method was developed by the Polish mathematician Adam Adamandy Kochański [3]. His construction results with π estimation equals to 3.141533…. Kochański's geometrical construction can be done with only one opening of a compass. In this case the process is not iterative. It is one-time construction.

Table 1. Estimated values of the number pi

N	a=N*sin(x)	b=N*tan(x)	c=a+(b-a)/2	d=a+(b-a)/3
6	3.00000	3.46410	3.23205	3.15470
12	3.10583	3.21539	3.16061	3.14235
24	3.12567	3.17389	3.14978	3.14174
48	3.13263	3.15966	3.14614	3.14164
96	3.13585	3.15313	3.14449	3.14161

2. MATERIAL AND METHODS

For our purpose, we consider two basic methods, Snells' rectification method and Dörrie's method [2,4]. Both methods were developed to accelerate Archimedes' process. Here, we are going the next step further. Our two approaches use the values generated by Snell's and Dörrie's method to construct better approximations for the number π. We listed all used methods in this work in Table 2. In our notation we added X (after M) to indicate that the method (M) is the result of combinations. We assumed that combination occurred when the composite method is defined by elements already calculated in its components, [5-7]. Consider three of the following methods: MX4: Snell-P based on perimeter (P) of the circle, MX5: Snell-A based on area (A) of the circle, (Huygens, 1654)) and MX6: Ch-H based on the methods M1, M2 and M3, [5]. Table 2 represents the applied methods, their short descriptions, and the results for using them with N=3 and 6. (π=3.14159265358979...). The method M8 was invented by Cusanus (XV), Snell-Huygens (XVII), and again by Dörrie (XX century). One of the results of this presentation is detection that one Dörrie's formula (for B) was already known in XV and XVII centuries.

2.1 Algorithm 1: Snell's Rectification

Cardinal Nicolaus Cusanus (1401-1464) has elaborated the following rectification of the arc in the circle for the corresponding angle x: *arc* = 3*sin*(*x*)/(2 + *cos*(*x*)). It corresponds to the first convergent of the continued fraction for *sin*(*x*)/*x*. This formula was once again proposed two hundred years later by the Dutch mathematician and physicist Snell (Willebrord Snellius, 1580-1626). We do not know whether it was an original invention or used the known result obtained by the cardinal. Snell developed two bounds for the length of the arc, lower (M8: Snell-ArcL) and upper (M7: Snell-ArcU), Huygens (1654). We combine these two methods to define a better approximation (MX11; Szyszkowicz, 2015, [6]). To develop such an approach, we used Taylor series for the corresponding methods (Tables 3 and 4), in this case M7 and M8, and generated the new method as MX11=u*M7+v*M8. The coefficients u and v are determined by the following system of equations (see Table 4) to improve its accuracy: $u + v = 1, u/1620 - v/180=0$. The solution allows us to define a more accurate method of the form MX11=M7+(M8-M7)/10. Table 4 shows that in its Taylor series the next term after x is x to the power 7. We keep the element x (x to the power one) but eliminate x to the power 5. Here x = π/N and as N is growing N*MX11 goes to π.

The Ch-H (MX6, [5]) method can be derived differently than originally presented by its authors. The method can be determined (Richardson's approach) as the results of a linear combination $MX6 = a * M1 + b * M2 + c * M3$. Here we use all methods related to Archimedes' technique. We are able to improve the accuracy without additional calculations (increasing N). Using their Taylor representation, it is possible to keep the term with x (we need to satisfy the condition a+b+c=1) and to eliminate the terms

Table 2. Methods, their descriptions and the results for pi using N=3 and 6. Method M8 was invented by Cusanus, Snell-Huygens, and Dörrie

Method (X combined)	Description	N=3	N=6
M1, side, inscribed	sin(x)	2.598076	3.000000
M2, side, area circumscribed	tan(x)	5.196152	3.464101
M3, area, inscribed	sin(2x)/2	1.299038	2.598076
MX4=M1+(M2-M1)/3	Snell-P	3.464101	3.154700
MX5=M2+(M3-M2)/3	Snell-A	3.897114	3.175426
MX6=(32M1+4M2-6M3)/30	Ch-H	3.204293	3.142264
M7=(2 cos (x/3)+1) tan (x/3)	Snell-ArcU	3.144031	3.141740
M8=3 sin(x)/(2+cos (x))	Snell-ArcL	3.117691	3.140237
MX9=$(M2*M1*M1)^{1/3}$	A- Dörrie	3.273370	3.147345
MX10=M8+(MX9-M8)/5	Szysz-Dörrie	3.148827	3.141658
MX11=M7+(M8-M7)/10	Szyszkowicz	3.141397	3.141589

Table 3. Taylor series of the methods related to Archimedes' algorithm

Method	Taylor series
M1: $\sin(x)$	$x-\frac{x^3}{6}+\frac{x^5}{120}-\frac{x^7}{5040}+\frac{x^9}{362880}-\frac{x^{11}}{39916800}+O(x^{12})$
M2: $\tan(x)$	$x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}+\frac{62x^9}{2835}+\frac{1382x^{11}}{155925}+O(x^{12})$
M3: $\sin(2x)/2$	$x-\frac{2x^3}{3}+\frac{2x^5}{15}-\frac{4x^7}{315}+\frac{2x^9}{2835}-\frac{4x^{11}}{155925}+O(x^{12})$

Table 4. Taylor series of the presented methods

Method	Taylor series or analytic formula
M7	$x+\frac{x^5}{1620}+\frac{x^7}{40824}+\frac{7x^9}{6298560}+\frac{3931x^{11}}{78568237440}+O(x^{12})$
M8	$B=\frac{3ab}{2a+b}=\frac{3\tan(x)\sin(x)}{2\tan(x)+\sin(x)}=\frac{3\sin(x)}{2+\cos(x)}$
M8	$x-\frac{x^5}{180}-\frac{x^7}{1512}-\frac{x^9}{25920}+\frac{x^{11}}{391680}+O(x^{13})$
MX9	$A=\sqrt[3]{ab^2}=\sqrt[3]{\tan(x)\sin(x)^2}$
MX9	$x+\frac{x^5}{45}+\frac{4x^7}{567}+\frac{x^9}{405}+\frac{248x^{11}}{280665}+O(x^{13})$
MX10	$x+\frac{4x^7}{1134}+\frac{x^9}{2160}+\frac{227x^{11}}{1283040}+O(x^{13})$
MX11	$x-\frac{x^7}{22680}-\frac{x^9}{349920}+\frac{437x^{11}}{6235574400}+O(x^{13})$

with x with powers 3 and 5. This produces the following conditions on the coefficients $a+b+c=1, -\frac{a}{6}+\frac{b}{3}-\frac{2c}{3}=0, \frac{a}{120}+\frac{2}{15}b+\frac{2}{15}c=0$. The obtained linear system is easy to solve. The system results in the following formula $MX6 = (32M1 + 4M2 - 6M3)/30$. With the new set of the parameters a and b, this method is also defined as $MX6 = a * MX2 + b * MX4$, with the following conditions: $a+b=1, \frac{a}{20}+\frac{2b}{15}=0$ (see Tables 3 and 4).

2.2 Algorithm 2: Dörrie's Sequence

In his book, the German mathematician Heinrich Dörrie, in problem No. 38 presented another approach to improve Archimedes' method, [4]. He constructed two new series B and A, (the [B, A] interval) which give a better approximation for the length of the circumference (C) of the circle. For given values b and a (the [b, a] interval) are generated $B=\frac{3ab}{2a+b}$ and $A=\sqrt[3]{ab^2}$. He proved that the following inequalities hold b < B < C < A < a. The sequence of Bs increases to C, and the sequence of As decreases to C. Always the interval [b, a] contains the interval [B, A]. For example, starting with a regular hexagon d = 1, a = 2√3, b = 3 we have the following values from Dörrie's method: B = 3.140237343, A = 3.14734519, a precision achieved by Archimedes' method first with a 96-gon. It is interesting that the method used to generate the sequence B is the same formula as proposed by the cardinal Cusanus and by Snell (M8); see also Tables 2 and 4, and Fig. 1. In a similar way as the method MX11 was obtained, the method MX10 was derived. The method is constructed as follows: MX10=M8+(MX9-M8)/5.

3. RESULTS AND DISCUSSION

The program in R is presented below. It realizes some of the discussed methods. Finally, the results are given for N=64. The listing of this program allows for a better understanding of the presented material and the realized formulae. The program starts with a square (N=4) or hexagon (N=6).

```
#Program realizes the following methods: M1, M2, M8, MX9, and MX10.
options(digits=15)
N=4; b=2*sqrt(2); a=4 #square
N=6; b=3; a=2*sqrt(3) #hexagon
for (k in 1:5){
cn=c(k-1,N); print(cn)
arch = c(b,a) #Archimedes' results
# Dörrie:
B=3*a*b/(2*a + b)
A=(a*b*b)^(1/3)
dor = c(B,A) # Dörrie's results
#Szyszkowicz
S=B+(A-B)/5 # Szyszkowicz's method
res=c(arch,dor,S)
print(res)
#Next Archimedes:
a=2*a*b/(a+b)
b=sqrt(a*b)
N=N+N}
method=c("M1","M2","M8","MX9","MX10")
print(method)
print(pi); #The end
#The results for 96-gon
M1: 3.14103195089051; M2: 3.14271459964537; M8: 3.14159263357057
MX9: 3.14159273368372; MX10: 3.14159265359320; pi: 3.14159265358979
```

The main results of this paper are the two methods (MX10 and MX11), where we used Taylor series to justify their correctness and accuracy. The methods are very easy to program. Some calculations were executed as presented in the program in R. Table 5 shows the results for the Pfaff-Borchardt-Schwab algorithm (a, b values), Dörrie's method (A, B values) and the method MX10 proposed in this paper.

Table 6 presents the obtained results for the method MX11 and a few other methods already known in literature.

Method MX11 has an interesting geometrical interpretation and one example is presented here. Fig. 1 shows the rectification process for the arc corresponding to the angle x = 135 degrees. It is a relatively large angle and consequently the estimation is not very accurate. In this approach, we have to realize two methods, M7 and M8, to obtain lower and upper bounds for the length of the arc. Their arithmetic average is less accurate than the one generated by the MX11 method. We have already the value 3.11582354 for pi. The exact value for the length of this arc is $\frac{3}{4}\boldsymbol{\pi}$. It allows us to determine our accuracy obtained for the angle x=135 degrees using the MX11 method.

As the method needs also the angle x/3, we need to be able to do the trisection of a given angle x. In this case, it is possible to do this by a pure geometrical construction. It is easy to obtain the angle x/3. It's by using a half of the right angle (90/2 = 45 = 135/3). The lower (L) and upper (U) estimations are generated by the methods M8 and M7, respectively. They have geometrical interpretations: the angle's vertex has the distance r (radius) to the circle for L, and to the cutting point on the circumference for the angle x. We are using the method MX11 to obtain a better approximation for the number π.

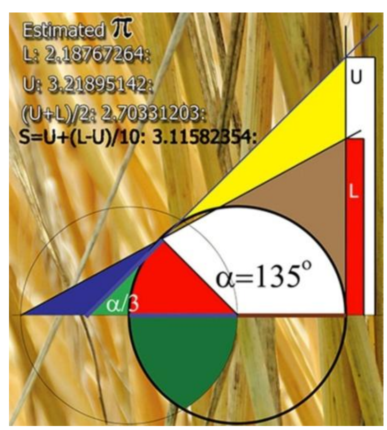

Fig. 1. Rectification of the arc - Szyszkowicz's method (MX11)

Table 5. The approximations generated by Archimedes, Dörrie's method, and Szyszkowicz's method (MX10)

Size	Archimedes		Dörrie		Szyszkowicz
N	**M1 (b)**	**M2 (a)**	**M8 (B)**	**MX9 (A)**	**M10 (B+(A-B)/5)**
6	3.0000000	3.4641016	3.1402373	3.1473452	3.1416589
12	3.1058285	3.2153903	3.1415100	3.1419279	3.1415936
24	3.1326286	3.1596599	3.1415875	3.1416133	3.1415927
48	3.1393502	3.1460862	3.1415923	3.1415939	3.1415927
96	3.1410320	3.1427146	3.1415926	3.1415927	3.1415927

Table 6. Estimated value of pi from various methods and N

Size	MX5	MX6	M7	M8	MX11
N	**Snell-A**	**Ch-H**	**Snell-ArcU**	**Snell-ArcL**	**Szyszkowicz**
6	3.89711432	3.20429399	3.14403156	3.11769145	3.14139755
8	3.33333333	3.15032227	3.14234913	3.13444650	3.14155887
10	3.21435552	3.14368811	3.14189972	3.13874170	3.14158392
12	3.17542648	3.14226497	3.14174002	3.14023734	3.14158975
14	3.15948495	3.14185286	3.14167196	3.14086739	3.14159151
16	3.15194804	3.14170766	3.14163906	3.14116990	3.14159214
18	3.14800282	3.14164881	3.14162159	3.14132974	3.14159240
20	3.14577340	3.14162228	3.14161162	3.14142063	3.14159252
22	3.14443578	3.14160929	3.14160560	3.14147540	3.14159258
24	3.14359354	3.14160249	3.14160179	3.14150999	3.14159261

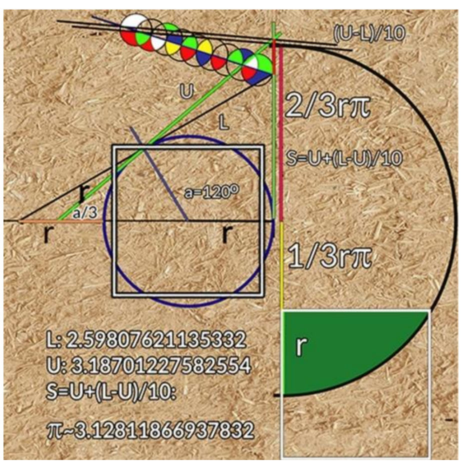

Fig. 2. Quadrature based on the rectification of the arc - Szyszkowicz's method (MX11)

Fig. 2 shows a more difficult situation. The angle of 120 degrees can't be trisected. We need the angle of 40 degrees. We may use other sources for such an angle, but not from a pure geometrical construction process. In this case, agraphic software was asked to rotate horizontal segment by 40 degrees. The method MX11 is applied and determines the segment S = U + (L − U)/10. Here, the main problem (mainly construction) is to determine the segment (U-L)/10. In Fig. 2, a series of small circles was used to realize the division into 10 equal parts. Thales' approach to divide a segment in a proportion is applied. The obtained segment (2/3πr) is extended by 1/3πr and r. It allows us to perform the squaring of the rectangle (interpreted as such) of sides πr and r. Consequently, we approximated the quadrature of our circle with an estimated value of the number π. In the geometrical process Thales theorem on proportion is applied to divide the segment U-L into 10 equal parts.

4. CONCLUSION

The illustrative results summarized obtained approximations by various methods. As the values show, the best approximation is produced by the MX10 method. The method is the result of the combination of two sequences generated by Dörrie's algorithm.

Well known methods to approximate the number pi are realized. The Taylor series of these methods (and Richardson's extrapolation) allow producing new methods with better convergence properties. As the main results, two methods are proposed: (i) combined Dörrie's sequence (MX10 method), (ii) combined Snell's sequence (MX11).

Two methods presented here improve Archimedes' technique. The method MX11 can be used geometrically for an angle x, if x/3 can be constructed to execute an approximate quadrature of the circle. In addition, the presented methodology has an educational aspect.

COMPETING INTERESTS

Author has declared that no competing interests exist.

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