



# Alternative Proof of Linear Tangle and Linear Obstacle: An Equivalence Result

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*Author's contribution*

*The sole author designed, analysed, interpreted and prepared the manuscript.*

*Article Information*

DOI: 10.9734/ARJOM/2023/v19i8687

**Open Peer Review History:**

This journal follows the Advanced Open Peer Review policy. Identity of the Reviewers, Editor(s) and additional Reviewers, peer review comments, different versions of the manuscript, comments of the editors, etc are available here: <https://www.sdiarticle5.com/review-history/99574>

**Received: 08/03/2023**

**Accepted: 11/05/2023**

**Published: 24/05/2023**

**Original Research Article**

## Abstract

Linear-width is a widely recognized and highly valued graph width parameter. The concepts of linear tangle and linear obstacle are dual concepts of linear-width. In this concise paper, we present an alternative proof of the equivalence between linear tangle and linear obstacle.

*Keywords: Linear width; linear tangle; linear obstacle, connectivity system.*

## 1 Introduction

The examination of width parameters holds significant importance in the fields of graph theory and combinatorics, as evidenced by the numerous publications on this topic (e.g., [1-13,14,15]). Linear width, a parameter of graph theory, has been studied in several papers [16,17,13].

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Linear tangle, a concept initially introduced in reference [18], plays a critical role in determining whether a linear width is at most  $k$  ( $k$  is a natural number), where  $k+1$  represents the order of the tangle (also see references [16,17,3]).

Graph searching games are well-known in the fields of graph theory and game theory. A strategy in which the player always emerges victorious is termed a "winning strategy". This winning strategy is characterized by various width parameters and their dual concepts commonly employed in graph theory (e.g., [3,19,20,21]). For instance, the concept of  $(k, m)$ -obstacle on connectivity systems is proposed in reference [3].

In this succinct paper, we present a direct proof for the equivalence between linear tangles and  $(k, 1)$ -obstacle, which are referred to as linear obstacle in this paper.

## 2 Definitions and Notations in this Paper

In this section, we provide mathematical definitions for each concept.

**Symmetric Submodular Function:** The definition of a symmetric submodular function is given below.

**Definition 1:** Let  $X$  be a finite set. A function  $f: X \rightarrow \mathbb{N}$  is called symmetric submodular if it satisfies the following conditions:

- $\forall A \subseteq X, f(A) = f(X \setminus A)$ .
- $\forall A, B \subseteq X, f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$ .

In this short paper, a pair  $(X, f)$  of a finite set  $X$  and a symmetric submodular function  $f$  is called a connectivity system. In this paper, we use the notation  $f$  for a symmetric submodular function, a finite set  $X$ , and a natural number  $k, m$ . A set  $X$  is  $k$ -efficient if  $f(X) \leq k$ .

**Linear tangle:** The definition of a linear tangle is given below.

**Definition 2 [18]:** Let  $X$  be a finite set and  $f$  be a symmetric submodular function. A linear tangle of order  $k+1$  on a connectivity system  $(X, f)$  is a family  $L$  of  $k$ -efficient subsets of  $X$ , satisfying the following axioms:

- (L1)  $\emptyset \in L$ ,
- (L2) For each  $k$ -efficient subset  $A$  of  $X$ , exactly one of  $A$  or  $X \setminus A$  in  $L$ ,
- (L3) If  $A, B \in L, e \in X$ , and  $f(\{e\}) \leq k$ , then  $A \cup B \cup \{e\} \neq X$  holds.

**Linear obstacle: Deep relation to  $(k, m)$ -obstacle:** The definition of  $(k, m)$ -obstacle is shown below.

**Definition 3 [4]:** Let  $X$  be a finite set and  $f$  be a symmetric submodular function. In a connectivity system  $(X, f)$ , the set family  $O \subseteq 2^X$  is called a  $(k, m)$ -obstacle if the following axioms hold true:

- (O1)  $A \in O, f(A) \leq k$ ,
- (O2)  $A \subseteq B \subseteq X, B \in O, f(A) \leq k \Rightarrow A \in O$ ,
- (O3)  $A, B, C \subseteq X, A \cup B \cup C = X, A \cap B = \emptyset, f(A) \leq k, f(B) \leq k, |C| \leq m \Rightarrow$  either  $A \in O$  or  $B \in O$ .

This paper deals with  $(k, m)$ -obstacle for the case where  $m = 1$ . In this article, we call  $(k, 1)$ -obstacle "Linear obstacle" of order  $k+1$ . Therefore, a linear obstacle is defined as follows:

**Definition 4:** Let  $X$  be a finite set and  $f$  be a symmetric submodular function. In a connectivity system  $(X, f)$ , the set family  $O \subseteq 2^X$  is called a linear obstacle of order  $k + 1$  if the following axioms hold true:

$$(O1) A \in O, f(A) \leq k,$$

$$(O2) A \subseteq B \subseteq X, B \in O, f(A) \leq k \Rightarrow A \in O,$$

$$(O3) A, B, C \subseteq X, A \cup B \cup C = X, A \cap B = \emptyset, f(A) \leq k, f(B) \leq k, |C| \leq 1 \Rightarrow \text{either } A \in O \text{ or } B \in O.$$

### 3 Equivalence between Linear Tangle and Linear Obstacle

The main result of this paper is below.

**Theorem 1.** Let  $X$  be a finite set and  $f$  be a symmetric submodular function. Under the assumption that  $f(\{e\}) \leq k$  for every  $e \in X$ ,  $F$  is a linear tangle of order  $k+1$  on  $(X, f)$  iff  $F$  is a linear obstacle of order  $k+1$  on  $(X, f)$ .

#### Proof of Theorem 1:

##### Step 1:

Assume  $F$  is a linear tangle of order  $k+1$  on connectivity system  $(X, f)$ . We need to show that  $F$  satisfies the axioms of a linear obstacle of order  $k+1$ .

We show that axiom (O1) holds. From the definition of a linear tangle,  $F$  contains  $k$ -efficient subsets of  $X$ . By the assumption that  $f(\{e\}) \leq k$  for every  $e \in X$ , any set  $A$  in  $F$  satisfies  $f(A) \leq k$ , which is consistent with the requirement in (O1).

Next, we show that axiom (O2) holds. Let  $A \subseteq B \subseteq X$  such that  $B \in F$  and  $f(A) \leq k$ . We know from the linear tangle definition that exactly one of  $B$  or  $X \setminus B$  is in  $F$ . If  $A=B$ , then  $A \in F$ . If  $A \neq B$ , then  $X \setminus A \subseteq X \setminus B$ . Since  $X \setminus B \in F$  and  $f(X \setminus A) = f(A) \leq k$ ,  $A \in F$ . Thus, axiom (O2) holds for  $F$ .

Finally, we show that axiom (O3) holds. To demonstrate axiom (O3), we will use proof by contradiction. Assume that axiom (O3) does not hold, which means either  $A \notin L$  and  $B \notin L$  or  $A \in L$  and  $B \in L$ . We will consider both cases.

Case 1:  $A \notin L$  and  $B \notin L$

Since  $f(B) \leq k$ , by axiom (L2),  $B \in L$ . As  $A \cap B = \emptyset$ , we have  $A \subseteq B$ . Since  $f(A) \leq k$ , by the previously demonstrated axiom (O2),  $A \in L$ , which leads to a contradiction.

Case 2:  $A \in L$  and  $B \in L$

In this case, there is a contradiction with axiom (L3).

Thus, in both cases, we arrive at contradictions, which means axiom (O3) must hold.

##### Step 2:

Assume  $F$  is a linear obstacle of order  $k+1$  on  $(X, f)$ . We need to show that  $F$  satisfies the axioms of a linear tangle of order  $k+1$ .

We show that axiom (L1) holds. Since  $f(\emptyset) = f(X) \leq k$ ,  $\emptyset$  is  $k$ -efficient, and by axiom (O1),  $\emptyset \in F$ .

Next, we show that axiom (L2) holds. Let  $A$  be a  $k$ -efficient subset of  $X$ . Since  $f(A) \leq k$  and  $f(X \setminus A) = f(A) \leq k$ , either  $A \in F$  or  $X \setminus A \in F$  by axiom (O3) with  $B=A$ ,  $C=\emptyset$ . If both  $A$  and  $X \setminus A$  were in  $F$ , it would contradict (O3) with  $B=X \setminus A$  and  $C=\emptyset$ . Hence, exactly one of  $A$  or  $X \setminus A$  is in  $F$ .

Finally, we show that axiom (L3) holds. Now, consider the three sets  $A$ ,  $B$ , and  $C$ . We have:

1.  $A \cup B \cup C = X$ ,
2.  $A \cap B = \emptyset$  (since  $A$  and  $B$  are both in  $F$  and  $F$  is a linear obstacle),
3.  $f(A) \leq k, f(B) \leq k$  (as  $A, B \in F$  and  $F$  is a linear obstacle),
4.  $|C| = 1$ .

From the definition of a linear obstacle, either  $A \in F$  or  $B \in F$  must hold. Since  $A, B \in F$  by assumption, this does not contradict the definition. Therefore,  $A \cup B \cup \{e\} \neq X$ , satisfying axiom (L3).

We have now proven both directions of Theorem 1:  $F$  is a linear tangle of order  $k+1$  on  $(X, f)$  if and only if  $F$  is a linear obstacle of order  $k+1$  on  $(X, f)$ . This proof is completed.

## 4 Conclusion and Future Tasks

In this succinct paper, we presented a direct proof for the equivalence between linear tangles and linear obstacle.

In the world of logic, the concept of a weak ideal is known [22,23]. This is a concept where some of the axioms of an ideal are replaced with weaker ones.

The definition below is based on the concept of an ideal on connectivity system  $(X, f)$  as defined in literature [1], and is an extension of the definition of weak ideal using the underlying set and power set, as defined in literature [18]. And an ideal on connectivity system  $(X, f)$  is dual concept of branch-width.

Our future goal is to investigate the relationship between weak ideal and various graph parameters and linear tangle. We also plan to study the dual concept of weak ideal, which is the concept of a weak filter.

**Definition 5 :** Let  $X$  be a finite set and  $f$  be a symmetric submodular function. In a connectivity system  $(X, f)$ , the set family  $W \subseteq 2^X$  is called a weak ideal of order  $k+1$  if the following axioms hold true:

- (IB) For every  $A \in W, f(A) \leq k$ .
- (IH) If  $A, B \subseteq X, f(A) \leq k$ ,  $A$  is a proper subset of  $B$  and  $B$  belongs to  $W$ , then  $A$  belongs to  $W$ .
- (WIS) If  $A$  belongs to  $W, B$  belongs to  $W$  and  $f(A \cup B) \leq k$ , then  $A \cup B \neq X$ .
- (IW)  $X$  does not belong to  $W$ .

## Acknowledgements

I humbly express my sincere gratitude to all those who have extended their invaluable support, enabling me to successfully accomplish this paper.

## Competing Interests

Author has declared that no competing interests exist.

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