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Determinants and Inverses of Fibonacci and Lucas Skew Symmetric Toeplitz Matrices

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Authors' contributions

This work was carried out in collaboration between all authors. Author ZJ designed the study, proposed the concerned problem. Authors XC and JW managed the analyses of the study and wrote the first draft of the manuscript. All authors read and approved the final manuscript.

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Abstract

In this paper, determinants and inverses of Fibonacci and Lucas skew symmetric Toeplitz matrices are given by constructing the special transformation matrices.

Keywords: Toeplitz matrix; determinant; inverse; Fibonacci number; Lucas number.

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1 Introduction

Toeplitz matrices have important applications in various disciplines including image processing, signal processing, and solving least squares problems [1, 2]. It is an ideal research area and hot topic for the inverses of Toeplitz matrices and the special matrices with famous numbers. Due to the special structre, it is desirable that the inversion of Toeplitz matrices can be reconstructed by use of a low number of its columns and the entries of the original Toeplitz matrix. The first result was presented by Trench [3] from its first and last columns of T^{-1}, on condition that the first element in the first column is not zero. The Trench's algorithm for the inversion of Toeplitz matrices is shown with a detailed proof in [4]. The authors [5] observed that if the last component of the first column is not zero, then T^{-1} can be recovered from its first and second columns. Labahn [6] proposed that formulas for the inverse of layered or striped Toeplitz matrices in terms of solutions of standard equations. Citations of a large number of results have been made in a series of papers and in the monographs Iohvidov [7] and Heining and Rost [8].

In addition, some scholars showed the explicit determinants and inverses of the special matrices involving famous numbers. In [9], M. Akbulak and D. Bozkurt treated originally Fibonacci and Lucas Toeplitz matrices with entries from Fibonacci and Lucas numbers, and they gave the upper and lower bounds for the spectral norms of the Fibonacci and Lucas Toeplitz matrix. Jaiswal [10] evaluated some determinants of circulant matrices whose components are the generalized Fibonacci numbers. Lind considered the determinants of circulant and skew-circulant matrices involving Fibonacci numbers in [11]. Lin showed the determinant of the Fibonacci-Lucas quasi-cyclic matrices in [12]. Circulant matrices with Fibonacci and Lucas numbers are discussed and their explicit determinants and inverses are proposed in [13]. The authors provided determinants and inverses of circulant matrices with Jacobsthal and Jacobsthal-Lucas numbers in [14]. The explicit determinants of circulant and left circulant matrices including Tribonacci numbers and generalized Lucas numbers are shown based on Tribonacci numbers and generalized Lucas numbers in [15]. In [16], circulant type matrices with the k-Fibonacci and k-Lucas numbers are considered and the explicit determinants and inverse matrices are presented by constructing the transformation matrices. Jiang et al. [17] gave the invertibility of circulant type matrices with the sum and product of Fibonacci and Lucas numbers and provided the determinants and the inverses of these matrices. And for the RSFPLR circulant matrices and the RSLPFL circulant matrices involving Padovan, Perrin, Tribonacci and the generalized Lucas numbers by the inverse factorization of polynomial in [18]. It should be noted that Jiang and Zhou [19] obtained the explicit formula for spectral norm of an r-circulant matrix whose entries in the first row are alternately positive and negative, and the authors [20] investigated explicit formulas of spectral norms for g-circulant matrices with Fibonacci and Lucas numbers. The authors [21] proposed the invertibility criterium of the generalized Lucas skew circulant type matrices and provided their determinants and the inverse matrices. Furthermore, in [22] the determinants and inverses are discussed and evaluated for Tribonacci skew circulant type matrices.

In this paper, we will show the recursive formulas of determinants and inverses of the Fibonacci skew symmetric Toeplitz matrices involving Fibonacci numbers, and Lucas skew symmetric Toeplitz matrices involving Lucas numbers.

Here the Fibonacci and Lucas sequences are defined by the following recursion relations, respectively:

$$\begin{array}{lllll} F_{n+1} = F_n + F_{n-1} & (n \geqslant 1) & \text{where} & F_0 = 0, & F_1 = 1, \\ L_{n+1} = L_n + L_{n-1} & (n \geqslant 1) & \text{where} & L_0 = 2, & L_1 = 1. \end{array}$$

Definition 1.1. An $n \times n$ Fibonacci skew symmetric Toeplitz matrix is meant a square matrix of the form as

$$\mathbf{T}_{F_n} = \begin{pmatrix} 0 & F_2 & F_3 & \cdots & F_{n-1} & F_n \\ -F_2 & 0 & F_2 & \cdots & F_{n-2} & F_{n-1} \\ -F_3 & -F_2 & 0 & \cdots & F_{n-3} & F_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -F_{n-1} & -F_{n-2} & -F_{n-3} & \cdots & 0 & F_2 \\ -F_n & -F_{n-1} & -F_{n-2} & \cdots & -F_2 & 0 \end{pmatrix}_{n\times n}, \tag{1.1}$$

where $F_2,\ F_3, \cdots,\ F_n$ are the Fibonacci numbers.

Obviously, this matrix is completely determined by its first row, and $\mathbf{T}_{F_n}^{\mathrm{T}} = -\mathbf{T}_{F_n}$.

Definition 1.2. An $n \times n$ Lucas skew symmetric Toeplitz matrix is meant a square matrix of the form as

$$\mathbf{T}_{L_n} = \begin{pmatrix} 0 & L_2 & L_3 & \cdots & L_{n-1} & L_n \\ -L_2 & 0 & L_2 & \cdots & L_{n-2} & L_{n-1} \\ -L_3 & -L_2 & 0 & \cdots & L_{n-3} & L_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -L_{n-1} & -L_{n-2} & -L_{n-3} & \cdots & 0 & L_2 \\ -L_n & -L_{n-1} & -L_{n-2} & \cdots & -L_2 & 0 \end{pmatrix}_{n\times n}, \tag{1.2}$$

where $L_2,\ L_3, \cdots,\ L_n$ are the Lucas numbers.

Obviously, this matrix is completely determined by its first row, and $\mathbf{T}_{L_n}^{\mathrm{T}} = -\mathbf{T}_{L_n}$.

Lemma 1.1. *Assume that $\alpha_1, \cdots, \alpha_n$ are all positive integers, then the matrix W_1 and W_1^{-1} are of the forms as*

$$W_1 = \begin{pmatrix} \alpha_1 & 0 & \cdots & \cdots & 0 \\ \alpha_2 & \alpha_1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \alpha_{n-1} & \ddots & \ddots & \ddots & 0 \\ \alpha_n & \alpha_{n-1} & \cdots & \alpha_2 & \alpha_1 \end{pmatrix}_{n\times n}, \quad W_1^{-1} = \begin{pmatrix} \beta_1 & 0 & \cdots & \cdots & 0 \\ \beta_2 & \beta_1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \beta_{n-1} & \ddots & \ddots & \ddots & 0 \\ \beta_n & \beta_{n-1} & \cdots & \beta_2 & \beta_1 \end{pmatrix}_{n\times n},$$

where

$$\beta_i = \frac{(-1)^{i-1} p_i}{(\alpha_1)^i}, \quad p_1 = 1, \quad p_2 = \alpha_2, \quad p_i = \alpha_2 p_{i-1} + \sum_{j=1}^{i-2} (-1)^j \alpha_1^j \alpha_{j+2} p_{i-1-j}, \ (i = 3, 4, 5, \cdots, n).$$

2 Determinant and Inverse of the Fibonacci Skew Symmetric Toeplitz Matrix

In this section, we will give the determinant and the inverse of the matrix $\mathbf{T}_{F_n}$.

As we know that if n is an odd number, the determinant of an n-dimension skew symmetric matrix is zero. So in this section we always assume that n is an even number.

Theorem 2.1. *Let $\mathbf{T}_{F_n}$ be a Fibonacci skew symmetric Toeplitz matrix as the form of (1.1), we have*

$$\det \mathbf{T}_{F_n} = F_n[a_1 \det \mathcal{C}_{n-2}([b_i]_{i=2}^{n-1}, -1, -1, 0, 1, 1) - b_1 \det \mathcal{D}_{n-2}([F_i]_{i=n-1}^{2}, -1, -1, 0, 1, 1)] \tag{2.1}$$

where

$$\mathcal{C}_{n-2}([b_i]_{i=2}^{n-1}, -1, -1, 0, 1, 1) =$$

$$\begin{pmatrix} b_2 & b_3 & b_4 & \cdots & \cdots & \cdots & b_{n-3} & b_{n-2} & b_{n-1} \\ 2 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 3 & 2 & 1 & \ddots & & & & & \vdots \\ -1 & 0 & 1 & 1 & \ddots & & & & \vdots \\ -1 & -1 & 0 & 1 & 1 & \ddots & & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & -1 & -1 & 0 & 1 & 1 \end{pmatrix}_{(n-2)\times(n-2)},$$

$$\mathcal{D}_{n-2}([F_i]_{i=n-1}^{2}, -1, -1, 0, 1, 1) =$$

$$\begin{pmatrix} F_{n-1} & F_{n-2} & F_{n-3} & \cdots & \cdots & \cdots & F_4 & F_3 & F_2 \\ 2 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 3 & 2 & 1 & \ddots & & & & & \vdots \\ -1 & 0 & 1 & 1 & \ddots & & & & \vdots \\ -1 & -1 & 0 & 1 & 1 & \ddots & & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & -1 & -1 & 0 & 1 & 1 \end{pmatrix}_{(n-2)\times(n-2)},$$

$$[b_i]_{i=2}^{n-1} = b_2, b_3, \cdots, b_{n-1}; \quad [F_i]_{i=n-1}^{2} = F_{n-1}, F_{n-2}, \cdots, F_2,$$

$$a_1 = \sum_{i=0}^{n-2} F_{i+2} y_{n-i-1}, \quad b_1 = \sum_{i=0}^{n-4} (-F_{n-i-2} + \frac{F_{n-i-1}F_{n-1}}{F_n}) y_{n-i-1} + \frac{F_{n-1}}{F_n} y_2 + y_1,$$

$$b_2 = \frac{F_{n-1}}{F_n}, \quad b_i = -F_{i-1} + \frac{F_i F_{n-1}}{F_n}, \ (i = 3, 4, \cdots, n-1), \quad y_1 = 1 \quad y_3 = y_2^2, \quad y_3 + 2y_2 + 3 = 0,$$

$$y_i = -2y_{i-1} - 3y_{i-2} - \sum_{j=1}^{i-3} 2F_{j+2} y_{i-j-2}, \ (i = 4, 5, 6, \cdots, n-1),$$

$$\det \mathcal{C}_j([b_i]_{i=2}^{j+1}, -1, -1, 0, 1, 1) =$$
$$(-1)^{j+1} b_{j+1} \det \mathcal{E}_{j-1}(-1, -1, 0, 1, 1) + \det \mathcal{C}_{j-1}([b_i]_{i=2}^{j}, -1, -1, 0, 1, 1), \tag{2.2}$$

$$\det \mathcal{D}_j([F_i]_{i=j+1}^{2}, -1, -1, 0, 1, 1) =$$
$$(-1)^{j+1} F_{n-j} \det \mathcal{E}_{j-1}(-1, -1, 0, 1, 1) + \det \mathcal{D}_{j-1}([F_i]_{i=j+1}^{3}, -1, -1, 0, 1, 1), \tag{2.3}$$

$$\mathcal{E}_i(-1,-1,0,1,1) = \begin{pmatrix} 2 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 3 & 2 & 1 & \ddots & & & & \vdots \\ -1 & 0 & 1 & 1 & \ddots & & & \vdots \\ -1 & -1 & 0 & 1 & 1 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & -1 & -1 & 0 & 1 \end{pmatrix}_{i\times i},$$

$$\det \mathcal{E}_i(-1,-1,0,1,1) = \det \mathcal{E}_{i-4}(-1,-1,0,1,1) - \det \mathcal{E}_{i-3}(-1,-1,0,1,1) + \det \mathcal{E}_{i-1}(-1,-1,0,1,1),$$
$$(i = 5, 6, \cdots, n-3), \tag{2.4}$$
$$\det \mathcal{E}_1(-1,-1,0,1,1) = 2, \quad \det \mathcal{E}_2(-1,-1,0,1,1) = 1, \tag{2.5}$$
$$\det \mathcal{E}_3(-1,-1,0,1,1) = 0, \quad \det \mathcal{E}_4(-1,-1,0,1,1) = -1. \tag{2.6}$$

Proof. Let $\mathbf{T}_{F_n}$ be an $n \times n$ Fibonacci skew symmetric Toeplitz matrix. In the case $n \geq 4$, let

$$\mathcal{A}_1 = \begin{pmatrix} 1 & & & & & & & 0 \\ & & & & & & & 1 \\ & & & & & & 1 & -\frac{F_{n-1}}{F_n} \\ & & & & & 1 & 1 & -1 \\ & & & & 1 & 1 & -1 & \\ & & & \iddots & \iddots & \iddots & & \\ & & 1 & 1 & -1 & & & \\ 0 & 1 & 1 & -1 & & & & \end{pmatrix}, \quad \mathcal{B}_1 = \begin{pmatrix} 1 & 0 & & & & & & 0 \\ & y_{n-1} & & & & & & 1 \\ & y_{n-2} & & & & & 1 & \\ & y_{n-3} & & & & 1 & & \\ & \vdots & & & \iddots & & & \\ & y_3 & & 1 & & & & \\ & y_2 & 1 & & & & & \\ & y_1 & & & & & & \end{pmatrix},$$

be two $n \times n$ matrices, which are invertible. And

$$y_1 = 1, \quad y_3 = y_2^2, \quad y_3 + 2y_2 + 3 = 0, y_i = -2y_{i-1} - 3y_{i-2} - \sum_{j=1}^{i-3} 2F_{j+2}y_{i-j-2} \ (i = 4, 5, 6, \cdots, n-1).$$

Multiplying $\mathbf{T}_{F_n}$ by $\mathcal{A}_1$ from the left, then multiplying $\mathcal{B}_1$ from the right, we obtain

$$\mathcal{A}_1 \mathbf{T}_{F_n} \mathcal{B}_1 = \begin{pmatrix} 0 & a_1 & F_{n-1} & F_{n-2} & F_{n-3} & \cdots & \cdots & F_4 & F_3 & F_2 \\ -F_n & a_2 & -F_2 & -F_3 & -F_4 & \cdots & \cdots & -F_{n-3} & -F_{n-2} & -F_{n-1} \\ 0 & b_1 & b_2 & b_3 & b_4 & \cdots & \cdots & b_{n-3} & b_{n-2} & b_{n-1} \\ \vdots & 0 & 2F_2 & F_1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & F_4 & 2F_2 & \ddots & \ddots & & & & \vdots \\ \vdots & \vdots & 2F_3 & F_4 & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & \vdots & 2F_4 & 2F_3 & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & 2F_{n-4} & 2F_{n-5} & 2F_{n-6} & \cdots & \cdots & 2F_2 & F_1 & 0 \\ 0 & 0 & 2F_{n-3} & 2F_{n-4} & 2F_{n-5} & \cdots & \cdots & F_4 & 2F_2 & F_1 \end{pmatrix},$$

where

$$a_1 = \sum_{i=0}^{n-2} F_{i+2} y_{n-i-1}, \quad a_2 = -\sum_{i=0}^{n-3} F_{n-i-1} y_{n-i-1},$$

$$b_1 = \sum_{i=0}^{n-4} (-F_{n-i-2} + \frac{F_{n-i-1}F_{n-1}}{F_n}) y_{n-i-1} + \frac{F_{n-1}}{F_n} y_2 + y_1,$$

$$b_2 = \frac{F_{n-1}}{F_n}, \quad b_i = -F_{i-1} + \frac{F_i F_{n-1}}{F_n}, \ (i = 3, 4, \cdots, n-1)$$

and from the last matrix we can easily get,

$$\det(\mathcal{A}_1 \mathbf{T}_{F_n} \mathcal{B}_1) = F_n \det M_{n-1}(a_1, [F_i]_{i=n-1}^{2}, [b_i]_{i=1}^{n-1}),$$

where

$$M_{n-1}(a_1, [F_i]_{i=n-1}^{2}, [b_i]_{i=1}^{n-1}) =$$

$$\begin{pmatrix}
a_1 & F_{n-1} & F_{n-2} & F_{n-3} & \cdots & \cdots & F_4 & F_3 & F_2 \\
b_1 & b_2 & b_3 & b_4 & \cdots & \cdots & b_{n-3} & b_{n-2} & b_{n-1} \\
0 & 2F_2 & F_1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & F_4 & 2F_2 & \ddots & \ddots & & & & \vdots \\
\vdots & 2F_3 & F_4 & \ddots & \ddots & \ddots & & & \vdots \\
\vdots & 2F_4 & 2F_3 & \ddots & \ddots & \ddots & \ddots & & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 2F_{n-4} & 2F_{n-5} & 2F_{n-6} & \cdots & \cdots & 2F_2 & F_1 & 0 \\
0 & 2F_{n-3} & 2F_{n-4} & 2F_{n-5} & \cdots & \cdots & F_4 & 2F_2 & F_1
\end{pmatrix}_{(n-1)\times(n-1)}.$$

In order to simply compute the determinant of $M_{n-1}(a_1, [F_i]_{i=n-1}^{2}, [b_i]_{i=1}^{n-1})$, we apply methods of elementary row transformation to this matrix, then we can obtain $M'_{n-1}(a_1, [F_i]_{i=n-1}^{2}, [b_i]_{i=1}^{n-1})$, it is of the form as,

$$M'_{n-1}(a_1, [F_i]_{i=n-1}^{2}, [b_i]_{i=1}^{n-1}) =$$

$$\begin{pmatrix}
a_1 & F_{n-1} & F_{n-2} & F_{n-3} & \cdots & \cdots & \cdots & F_4 & F_3 & F_2 \\
b_1 & b_2 & b_3 & b_4 & \cdots & \cdots & \cdots & b_{n-3} & b_{n-2} & b_{n-1} \\
0 & 2 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & 3 & 2 & 1 & \ddots & & & & & \vdots \\
\vdots & -1 & 0 & 1 & 1 & \ddots & & & & \vdots \\
\vdots & -1 & -1 & 0 & 1 & 1 & \ddots & & & \vdots \\
\vdots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & \cdots & \cdots & 0 & -1 & -1 & 0 & 1 & 1
\end{pmatrix},$$

then we can obtain

$$\begin{aligned}&\det M_{n-1}^{'}(a_1,[F_i]_{i=n-1}^{2},[b_i]_{i=1}^{n-1})=\\&\qquad a_1\det\mathcal{C}_{n-2}([b_i]_{i=2}^{n-1},-1,-1,0,1,1)-b_1\det\mathcal{D}_{n-2}([F_i]_{i=n-1}^{2},-1,-1,0,1,1),\end{aligned}$$

where $\mathcal{C}_{n-2}([b_i]_{i=2}^{n-1},-1,-1,0,1,1)$ and $\mathcal{D}_{n-2}([b_i]_{i=2}^{n-1},-1,-1,0,1,1)$ are the forms as in the interpretation of Theorem 2.1.

To obtain determinants of matrices $\mathcal{C}_{n-2}([b_i]_{i=2}^{n-1},-1,-1,0,1,1)$ and $\mathcal{D}_{n-2}([F_i]_{i=n-1}^{2},-1,-1,0,1,1)$, develop them in accordance with the last column, and in turn it, we can get recursive formulas (2.2)-(2.6). And observe that

$$\det\mathcal{A}_1=\det\mathcal{B}_1=(-1)^{\frac{(n-1)(n-2)}{2}},$$

we can obtain $\det\mathbf{T}_{F_n}$, which completes the proof. □

Theorem 2.2. *Let $\mathbf{T}_{F_n}$ be a Fibonacci skew symmetric Toeplitz matrix as the form of (1.1). If $\mathbf{T}_{F_n}$ is a nonsingular matrix, then*

$$\mathbf{T}_{F_n}^{-1}=\begin{pmatrix}0&\rho_{12}&\rho_{13}&\rho_{14}&\cdots&\rho_{1,n-1}&\rho_{1n}\\-\rho_{12}&0&\rho_{23}&\rho_{24}&\cdots&\rho_{2,n-1}&\rho_{1,n-1}\\-\rho_{13}&-\rho_{23}&0&\rho_{3,4}&\cdots&\rho_{2,n-2}&\rho_{1,n-2}\\-\rho_{14}&-\rho_{24}&-\rho_{3,4}&0&\cdots&\rho_{2,n-3}&\rho_{1,n-3}\\\vdots&\vdots&\vdots&\vdots&&\vdots&\vdots\\-\rho_{1,n-1}&-\rho_{2,n-1}&-\rho_{2,n-2}&-\rho_{2,n-3}&\cdots&0&\rho_{12}\\-\rho_{1,n}&-\rho_{1,n-1}&-\rho_{1,n-2}&-\rho_{1,n-3}&\cdots&-\rho_{12}&0\end{pmatrix},\qquad(2.7)$$

that is to say $\mathbf{T}_{F_n}^{-1}$ is skew symmetric about diagonal and symmetric about secondary diagonal as well, where

$$\rho_{12}=\delta_{1n},\quad\rho_{13}=\delta_{1,n-1}+\delta_{1n},\quad\rho_{23}=\delta_{2,n-1}+\delta_{2n},\quad\rho_{1n}=\delta_{11}-\frac{F_{n-1}}{F_n}\delta_{13}-\delta_{14},$$

$$\rho_{mj}=\delta_{m,n+2-j}+\delta_{m,n+3-j}-\delta_{m,n+4-j},\ (m=1,2,\cdots,n;j=4,5,\cdots,n-1),$$

$$\delta_{11}=-\frac{1}{F_n},\quad\delta_{12}=\frac{d_1}{a_1},\quad\delta_{13}=\frac{d_2}{\ell_1}+\sum_{i=1}^{n-3}d_{i+2}\xi_i^{'},\quad\delta_{1j}=d_2\eta_{j-3}^{'}+\sum_{i=1}^{n-3}d_{i+2}\sigma_{i,j-3},\ (j=4,5,\cdots,n),$$

$$\delta_{m1}=0,\ (m=2,3,\cdots,n),\qquad\delta_{m2}=\frac{y_{n+1-m}}{a_1},\ (m=2,3,\cdots,n),$$

$$\delta_{m3}=\frac{e_2y_{n+1-m}}{\ell_1}+\sum_{i=1}^{n-3}e_{i+2}y_{n+1-m}\xi_i^{'}+\xi_{n-1-m}^{'},\ (m=2,3,\cdots,n),$$

$$\delta_{mj}=e_2y_{n+1-m}\eta_{j-3}^{'}+\sum_{i=1}^{n-3}e_{i+2}y_{n+1-m}\sigma_{i,j-3}+\sigma_{n-1-m,j-3},\ (m=2,3,\cdots,n;\ j=4,5,\cdots,n),$$

$$d_1=\frac{a_2}{F_n},\quad d_i=\frac{-a_2F_{n-i+1}-a_1F_i}{a_1F_n},\ (i=2,3,\cdots,n-1),\quad e_i=\frac{-F_{n-i+1}}{a_1},\ (i=2,3,\cdots,n-1),$$

$$\ell_1=-\frac{b_1}{a_1}F_{n-1}+b_2-V_1W_1^{-1}U_1,\quad U_1=(2F_2,F_4,2F_3,\cdots,2F_{n-4},2F_{n-3})^T,$$

$$V_1=(-\frac{b_1}{a_1}F_{n-2}+b_3,-\frac{b_1}{a_1}F_{n-3}+b_4,\cdots,-\frac{b_1}{a_1}F_3+b_{n-2},-\frac{b_1}{a_1}F_2+b_{n-1}),$$

$$W_1^{-1}=(a_{i,j})_{i,j=1}^{n-3},\quad a_{i,j}=\begin{cases}\beta_{i-j+1},&i\geqslant j,\\0,&i<j,\end{cases}\quad(i,j=1,2,\cdots,n-3),\quad\beta_i=(-1)^{i-1}p_i,$$

$$p_1 = 1,\quad p_2 = 2,\quad p_i = 2p_{i-1} - 3p_{i-2} + \sum_{j=2}^{i-2}(-1)^j 2F_{j+1}p_{i-j-1},\ (i = 3, 4, 5, \cdots, n-3),$$

$$\eta_m^{'} = -\frac{1}{\ell_1}\eta_m,\ (m = 1, 2, \cdots, n-3),\quad \xi_m^{'} = -\frac{1}{\ell_1}\xi_m,\ (m = 1, 2, \cdots, n-3),$$

$$\eta_m = \sum_{i=m}^{n-3}(-\frac{b_1}{a_1}F_{n-1-i} + b_{i+2})\beta_{i+1-m},\ (m = 1, 2, \cdots, n-3),$$

$$\xi_1 = 2\beta_1,\quad \xi_2 = 2\beta_2 + 3\beta_1,\quad \xi_m = 2\beta_m + 3\beta_{m-1} + \sum_{i=1}^{m-2}\beta_{m-1-i}\cdot 2F_{i+2},\ (m = 3, 4, \cdots, n-3),$$

$$\sigma_{i,j} = \beta_{i-j+1} + \frac{1}{\ell_1}\xi_i\eta_j,\ (i, j = 1, 2, \cdots, n-3),$$

$a_1,\ a_2,\ b_i,\ y_i,\ (i = 1, 2, \cdots, n-1)$ *are the same as in* Theorem 2.1.

Proof. Let $\mathcal{A}_2$ and $\mathcal{B}_2$ be two $n \times n$ invertible matrices, defined by

$$\mathcal{A}_2 = \begin{pmatrix} 0 & 1 & & & & & \\ 1 & & & & & & \\ -\frac{b_1}{a_1} & 0 & 1 & & & & \\ & & & 1 & & & \\ & & & & \ddots & & \\ & & & & & 1 & \\ & & & & & & 1 \end{pmatrix}_{n\times n},$$

$$\mathcal{B}_2 = \begin{pmatrix} 1 & d_1 & d_2 & d_3 & \cdots & d_{n-2} & d_{n-1} \\ & 1 & e_2 & e_3 & \cdots & e_{n-2} & e_{n-1} \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & & & \ddots & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}_{n\times n},$$

where $d_1 = \frac{a_2}{F_n},\ d_i = \frac{-a_2F_{n-i+1}-a_1F_i}{a_1F_n},\ e_i = \frac{-F_{n-i+1}}{a_1},\ (i = 2, 3, \cdots, n-1).$

Let $\mathcal{A}_1$ and $\mathcal{B}_1$ be as in the proof of Theorem 2.1, multiplying $\mathcal{A}_1\mathbf{T}_{F_n}\mathcal{B}_1$ by $\mathcal{A}_2$ from the left and by $\mathcal{B}_2$ from the right, we obtain

$$\mathcal{A}_2\mathcal{A}_1\mathbf{T}_{F_n}\mathcal{B}_1\mathcal{B}_2$$
$$= \begin{pmatrix} -F_n & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & a_1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & -\frac{b_1}{a_1}F_{n-1} + b_2 & -\frac{b_1}{a_1}F_{n-2} + b_3 & \cdots & \cdots & -\frac{b_1}{a_1}F_3 + b_{n-2} & -\frac{b_1}{a_1}F_2 + b_{n-1} \\ \vdots & \vdots & 2F_2 & F_1 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & F_4 & 2F_2 & \ddots & \ddots & & \vdots \\ \vdots & \vdots & 2F_3 & F_4 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 2F_{n-3} & 2F_{n-4} & \cdots & F_4 & 2F_2 & F_1 \end{pmatrix},$$

this matrix admits a block partition of the form as

$$\mathcal{A}_2\mathcal{A}_1\mathbf{T}_{F_n}\mathcal{B}_1\mathcal{B}_2 = \mathbf{N} \oplus \mathbf{R},$$

where $\mathbf{N} \oplus \mathbf{R}$ is the direct sum of $\mathbf{N}$ and $\mathbf{R}$. $\mathbf{N} = \mathrm{diag}(-F_n,\ a_1)$ is a nonsingular diagonal matrix,

$$\mathbf{R} = \begin{pmatrix} -\frac{b_1}{a_1}F_{n-1}+b_2 & -\frac{b_1}{a_1}F_{n-2}+b_3 & -\frac{b_1}{a_1}F_{n-3}+b_4 & \cdots & -\frac{b_1}{a_1}F_3+b_{n-2} & -\frac{b_1}{a_1}F_2+b_{n-1} \\ 2F_2 & F_1 & 0 & \cdots & \cdots & 0 \\ F_4 & 2F_2 & F_1 & \ddots & & \vdots \\ 2F_3 & F_4 & 2F_2 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 2F_{n-3} & 2F_{n-4} & 2F_{n-5} & \cdots & 2F_2 & F_1 \end{pmatrix}.$$

Let $\mathcal{A} = \mathcal{A}_2\mathcal{A}_1$ and $\mathcal{B} = \mathcal{B}_1\mathcal{B}_2$, we can get,

$$\mathbf{T}_{F_n}^{-1} = \mathcal{B}(\mathbf{N}^{-1} \oplus \mathbf{R}^{-1})\mathcal{A},$$

where

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \\ 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ -\frac{b_1}{a_1} & \vdots & & & & \iddots & 1 & -\frac{F_{n-1}}{F_n} \\ 0 & \vdots & & & \iddots & 1 & 1 & -1 \\ \vdots & \vdots & & \iddots & \iddots & \iddots & \iddots & 0 \\ \vdots & \vdots & \iddots & \iddots & \iddots & \iddots & \iddots & \vdots \\ \vdots & 0 & 1 & 1 & -1 & \iddots & & \vdots \\ 0 & 1 & 1 & -1 & 0 & \cdots & \cdots & 0 \end{pmatrix},$$

and

$$\mathcal{B} = \begin{pmatrix} 1 & d_1 & d_2 & d_3 & \cdots & d_{n-3} & d_{n-2} & d_{n-1} \\ 0 & y_{n-1} & e_2y_{n-1} & e_3y_{n-1} & \cdots & e_{n-3}y_{n-1} & e_{n-2}y_{n-1} & e_{n-1}y_{n-1}+1 \\ 0 & y_{n-2} & e_2y_{n-2} & e_3y_{n-2} & \cdots & e_{n-3}y_{n-2} & e_{n-2}y_{n-2}+1 & e_{n-1}y_{n-2} \\ 0 & y_{n-3} & e_2y_{n-3} & e_3y_{n-3} & \cdots & e_{n-3}y_{n-3}+1 & e_{n-2}y_{n-3} & e_{n-1}y_{n-3} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & y_3 & e_2y_3 & e_3y_3+1 & \cdots & e_{n-3}y_3 & e_{n-2}y_3 & e_{n-1}y_3 \\ 0 & y_2 & e_2y_2+1 & e_3y_2 & \cdots & e_{n-3}y_2 & e_{n-2}y_2 & e_{n-1}y_2 \\ 0 & y_1 & e_2 & e_3 & \cdots & e_{n-3} & e_{n-2} & e_{n-1} \end{pmatrix}.$$

Observe that the inverse matrix of $\mathbf{N}$ is of the form as,

$$\mathbf{N}^{-1} = \mathrm{diag}(-F_n^{-1},\ a_1^{-1}).$$

Let $\mathbf{R} = \begin{pmatrix} -\frac{b_1}{a_1}F_{n-1}+b_2 & V_1 \\ U_1 & W_1 \end{pmatrix}$ be an $(n-2)\times(n-2)$ matrix, where $U_1 = (2F_2, F_4, 2F_3, \cdots,$

$2F_{n-4}, 2F_{n-3})^T$, $\quad V_1 = (-\frac{b_1}{a_1}F_{n-2} + b_3, -\frac{b_1}{a_1}F_{n-3} + b_4, \cdots, -\frac{b_1}{a_1}F_2 + b_{n-1})$,

$$W_1 = \begin{pmatrix} F_1 & & & & & & \\ 2F_2 & F_1 & & & & & \\ F_4 & 2F_2 & F_1 & & & & \\ 2F_3 & \ddots & \ddots & \ddots & & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & & \\ 2F_{n-5} & \ddots & \ddots & \ddots & \ddots & F_1 & \\ 2F_{n-4} & 2F_{n-5} & \cdots & 2F_3 & F_4 & 2F_2 & F_1 \end{pmatrix},$$

as $F_1 = 1$, so W_1 is an invertible matrix. Use Lemma 1.1 we can obtain the inverse matrix of W_1, $W_1^{-1} = (a_{i,j})_{i,j=1}^{n-3}$, as in the interpretation of Theorem 2.2.

Use Lemma 5 in [23], we obtain,

$$\mathbf{R}^{-1} = \begin{pmatrix} \frac{1}{\ell_1} & -\frac{1}{\ell_1}V_1W_1^{-1} \\ -\frac{1}{\ell_1}W_1^{-1}U_1 & W_1^{-1} + \frac{1}{\ell_1}W_1^{-1}U_1V_1W_1^{-1} \end{pmatrix},$$

where $\ell_1 = -\frac{b_1}{a_1}F_{n-1} + b_2 - V_1W_1^{-1}U_1$, and simply we can get $(\mathbf{N} \oplus \mathbf{R})^{-1}$,

$$(\mathbf{N} \oplus \mathbf{R})^{-1} = \begin{pmatrix} \frac{1}{-F_n} & 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & \frac{1}{a_1} & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \frac{1}{\ell_1} & \eta_1' & \eta_2' & \cdots & \cdots & \eta_{n-4}' & \eta_{n-3}' \\ 0 & 0 & \xi_1' & \sigma_{11} & \sigma_{12} & \cdots & \cdots & \sigma_{1,n-4} & \sigma_{1,n-3} \\ 0 & 0 & \xi_2' & \sigma_{21} & \sigma_{22} & \cdots & \cdots & \sigma_{2,n-4} & \sigma_{2,n-3} \\ 0 & 0 & \xi_3' & \sigma_{31} & \sigma_{32} & \cdots & \cdots & \sigma_{3,n-4} & \sigma_{3,n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & & \vdots & \vdots \\ 0 & 0 & \xi_{n-4}' & \sigma_{n-4,1} & \sigma_{n-4,2} & \cdots & \cdots & \sigma_{n-4,n-4} & \sigma_{n-4,n-3} \\ 0 & 0 & \xi_{n-3}' & \sigma_{n-3,1} & \sigma_{n-3,2} & \cdots & \cdots & \sigma_{n-3,n-4} & \sigma_{n-3,n-3} \end{pmatrix},$$

then multiplying $(\mathbf{N} \oplus \mathbf{R})^{-1}$ by $\mathcal{B}$ from the left, we can obtain

$$\mathcal{B}(\mathbf{N} \oplus \mathbf{R})^{-1} = \begin{pmatrix} \delta_{11} & \delta_{12} & \delta_{13} & \cdots & \delta_{1,n-1} & \delta_{1,n} \\ \delta_{21} & \delta_{22} & \delta_{23} & \cdots & \delta_{2,n-1} & \delta_{2,n} \\ \delta_{31} & \delta_{32} & \delta_{33} & \cdots & \delta_{3,n-1} & \delta_{3,n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \delta_{n-1,1} & \delta_{n-1,2} & \delta_{n-1,3} & \cdots & \delta_{n-1,n-1} & \delta_{n-1,n} \\ \delta_{n,1} & \delta_{n,2} & \delta_{n,3} & \cdots & \delta_{n,n-1} & \delta_{n,n} \end{pmatrix}.$$

In the end, we can obtain $\mathbf{T}_{F_n}^{-1}$,

$$\mathbf{T}_{F_n}^{-1} = \mathcal{B}(\mathbf{N} \oplus \mathbf{R})^{-1}\mathcal{A} =$$

$$\begin{pmatrix} 0 & \rho_{12} & \rho_{13} & \rho_{14} & \cdots & \rho_{1,n-1} & \rho_{1n} \\ -\rho_{12} & 0 & \rho_{23} & \rho_{24} & \cdots & \rho_{2,n-1} & \rho_{1,n-1} \\ -\rho_{13} & -\rho_{23} & 0 & \rho_{3,4} & \cdots & \rho_{2,n-2} & \rho_{1,n-2} \\ -\rho_{14} & -\rho_{24} & -\rho_{3,4} & 0 & \cdots & \rho_{2,n-3} & \rho_{1,n-3} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ -\rho_{1,n-1} & -\rho_{2,n-1} & -\rho_{2,n-2} & -\rho_{2,n-3} & \cdots & 0 & \rho_{12} \\ -\rho_{1,n} & -\rho_{1,n-1} & -\rho_{1,n-2} & -\rho_{1,n-3} & \cdots & -\rho_{12} & 0 \end{pmatrix},$$

where

$$\rho_{12} = \delta_{1n}, \quad \rho_{13} = \delta_{1,n-1} + \delta_{1n}, \quad \rho_{23} = \delta_{2,n-1} + \delta_{2n}, \quad \rho_{1n} = \delta_{11} - \frac{F_{n-1}}{F_n}\delta_{13} - \delta_{14},$$
$$\rho_{mj} = \delta_{m,n+2-j} + \delta_{m,n+3-j} - \delta_{m,n+4-j}, \ (m = 1, 2, \cdots, n; j = 4, 5, \cdots, n-1),$$

which completes the proof. □

3 Determinant and Inverse of the Lucas Skew Symmetric Toeplitz Matrix

In this section, we will give the determinant and the inverse of the matrix $\mathbf{T}_{L_n}$. And in this section we always assume that n is an even number, as well.

Theorem 3.1. *Let $\mathbf{T}_{L_n}$ be a Lucas skew symmetric Toeplitz matrix as the form of (1.2), we have*

$$\det \mathbf{T}_{L_n} = L_n[\tilde{a}_1 \det \tilde{\mathcal{C}}_{n-2}([\tilde{b}_i]_{i=2}^{n-1}, -1, -5, 0, 5, 1) - \tilde{b}_1 \det \tilde{\mathcal{D}}_{n-2}([L_i]_{i=n-1}^{2}, -1, -5, 0, 5, 1)] \tag{3.1}$$

where

$$\tilde{a}_1 = \sum_{i=0}^{n-2} L_{i+2}x_{n-i-1}, \quad \tilde{b}_1 = \sum_{i=0}^{n-4}(-L_{n-i-2} + \frac{L_{n-i-1}L_{n-1}}{L_n})x_{n-i-1} + \frac{3L_{n-1}}{L_n}x_2 + 3x_1,$$
$$\tilde{b}_2 = \frac{3L_{n-1}}{L_n}, \quad \tilde{b}_i = -L_{i-1} + \frac{L_iL_{n-1}}{L_n}, \ (i = 3, 4, \cdots, n-1), \quad x_1 = 1, \quad x_3 = x_2^2, \quad x_3 + 6x_2 + 7 = 0,$$
$$x_i = -6x_{i-1} - 7x_{i-2} - \sum_{j=1}^{i-3} 2L_{j+2}x_{i-j-2}, \ (i = 4, 5, 6, \cdots, n-1),$$

$$\tilde{\mathcal{C}}_{n-2}([\tilde{b}_i]_{i=2}^{n-1}, -1, -5, 0, 5, 1) =$$

$$\begin{pmatrix}
\tilde{b}_2 & \tilde{b}_3 & \tilde{b}_4 & \cdots & \cdots & \cdots & \tilde{b}_{n-3} & \tilde{b}_{n-2} & \tilde{b}_{n-1} \\
6 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
7 & 6 & 1 & \ddots & & & & & \vdots \\
-5 & 0 & 5 & 1 & \ddots & & & & \vdots \\
-1 & -5 & 0 & 5 & 1 & \ddots & & & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & -1 & -5 & 0 & 5 & 1
\end{pmatrix}_{(n-2)\times(n-2)},$$

$$\tilde{\mathcal{D}}_{n-2}([L_i]_{i=n-1}^{2}, -1, -5, 0, 5, 1) =$$

$$\begin{pmatrix} L_{n-1} & L_{n-2} & L_{n-3} & \cdots & \cdots & \cdots & L_4 & L_3 & L_2 \\ 6 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 7 & 6 & 1 & \ddots & & & & & \vdots \\ -5 & 0 & 5 & 1 & \ddots & & & & \vdots \\ -1 & -5 & 0 & 5 & 1 & \ddots & & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & -1 & -5 & 0 & 5 & 1 \end{pmatrix}_{(n-2)\times(n-2)},$$

$$[\tilde{b}_i]_{i=2}^{n-1} = \tilde{b}_2, \tilde{b}_3, \cdots, \tilde{b}_{n-1}; \quad [L_i]_{i=n-1}^{2} = L_{n-1}, L_{n-2}, \cdots, L_2,$$

$$\det \tilde{\mathcal{C}}_j([\tilde{b}_i]_{i=2}^{j+1}, -1, -5, 0, 5, 1) =$$
$$(-1)^{j+1} b_{j+1} \det \tilde{\mathcal{E}}_{j-1}(-1, -5, 0, 5, 1) + \det \tilde{\mathcal{C}}_{j-1}([\tilde{b}_i]_{i=2}^{j}, -1, -5, 0, 5, 1), \tag{3.2}$$

$$\det \tilde{\mathcal{D}}_j([L_i]_{i=j+1}^{2}, -1, -5, 0, 5, 1) =$$
$$(-1)^{j+1} L_{n-j} \det \tilde{\mathcal{E}}_{j-1}(-1, -5, 0, 5, 1) + \det \tilde{\mathcal{D}}_{j-1}([L_i]_{i=j+1}^{3}, -1, -5, 0, 5, 1), \tag{3.3}$$

$$\tilde{\mathcal{E}}_i(-1, -5, 0, 5, 1) = \begin{pmatrix} 6 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 7 & 6 & 1 & \ddots & & & & \vdots \\ -5 & 0 & 5 & 1 & \ddots & & & \vdots \\ -1 & -5 & 0 & 5 & 1 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & -1 & -5 & 0 & 5 \end{pmatrix}_{i\times i},$$

$$\det \tilde{\mathcal{E}}_i(-1, -5, 0, 5, 1) = \det \tilde{\mathcal{E}}_{i-4}(-1, -5, 0, 5, 1) - 5 \det \tilde{\mathcal{E}}_{i-3}(-1, -5, 0, 5, 1) + 5 \det \tilde{\mathcal{E}}_{i-1}(-1, -5, 0, 5, 1),$$
$$(i = 5, 6, \cdots, n-3), \tag{3.4}$$

$$\det \tilde{\mathcal{E}}_1(-1, -5, 0, 5, 1) = 6, \qquad \det \tilde{\mathcal{E}}_2(-1, -5, 0, 5, 1) = 29, \tag{3.5}$$

$$\det \tilde{\mathcal{E}}_3(-1, -5, 0, 5, 1) = 140, \quad \det \tilde{\mathcal{E}}_4(-1, -5, 0, 5, 1) = 671. \tag{3.6}$$

Proof. Let $\mathbf{T}_{L_n}$ be an $n \times n$ Lucas skew symmetric Toeplitz matrix. In the case $n \geq 4$, let

$$\Phi_1 = \begin{pmatrix} 1 & & & & & & & 0 \\ & & & & & & & 1 \\ & & & & & & 1 & -\frac{L_{n-1}}{L_n} \\ & & & & & 1 & 1 & -1 \\ & & & & 1 & 1 & -1 & \\ & & & \iddots & \iddots & \iddots & & \\ & & 1 & 1 & -1 & & & \\ 0 & 1 & 1 & -1 & & & & \end{pmatrix},$$

$$\Psi_1 = \begin{pmatrix} 1 & 0 & & & & & 0 \\ & x_{n-1} & & & & & 1 \\ & x_{n-2} & & & & 1 & \\ & x_{n-3} & & & 1 & & \\ & \vdots & & \iddots & & & \\ & x_3 & & 1 & & & \\ & x_2 & 1 & & & & \\ & x_1 & & & & & \end{pmatrix},$$

be two $n \times n$ matrices, which are invertible. And

$$x_1 = 1, \quad x_3 = x_2^2, \quad x_3 + 6x_2 + 7 = 0, \quad x_i = -6x_{i-1} - 7x_{i-2} - \sum_{j=1}^{i-3} 2L_{j+2}x_{i-j-2} \ (i = 4, 5, 6, \cdots, n-1).$$

Multiplying $\mathbf{T}_{L_n}$ by Φ_1 from the left, then multiplying Ψ_1 from the right, we obtain

$$\Phi_1 \mathbf{T}_{L_n} \Psi_1 = \begin{pmatrix} 0 & \tilde{a}_1 & L_{n-1} & L_{n-2} & L_{n-3} & \cdots & \cdots & L_4 & L_3 & L_2 \\ -L_n & \tilde{a}_2 & -L_2 & -L_3 & -L_4 & \cdots & \cdots & -L_{n-3} & -L_{n-2} & -L_{n-1} \\ 0 & \tilde{b}_1 & \tilde{b}_2 & \tilde{b}_3 & \tilde{b}_4 & \cdots & \cdots & \tilde{b}_{n-3} & \tilde{b}_{n-2} & \tilde{b}_{n-1} \\ \vdots & 0 & 2L_2 & L_1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & L_4 & 2L_2 & \ddots & \ddots & & & & \vdots \\ \vdots & \vdots & 2L_3 & L_4 & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & \vdots & 2L_4 & 2L_3 & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & 2L_{n-4} & 2L_{n-5} & 2L_{n-6} & \cdots & \cdots & 2L_2 & L_1 & 0 \\ 0 & 0 & 2L_{n-3} & 2L_{n-4} & 2L_{n-5} & \cdots & \cdots & L_4 & 2L_2 & L_1 \end{pmatrix},$$

where

$$\tilde{a}_1 = \sum_{i=0}^{n-2} L_{i+2}x_{n-i-1}, \quad \tilde{a}_2 = -\sum_{i=0}^{n-3} L_{n-i-1}x_{n-i-1},$$

$$\tilde{b}_1 = \sum_{i=0}^{n-4}(-L_{n-i-2} + \frac{L_{n-i-1}L_{n-1}}{L_n})x_{n-i-1} + \frac{3L_{n-1}}{L_n}x_2 + 3x_1,$$

$$\tilde{b}_2 = \frac{3L_{n-1}}{L_n}, \quad \tilde{b}_i = -L_{i-1} + \frac{L_i L_{n-1}}{L_n},\ (i = 3, 4, \cdots, n-1)$$

and from the last matrix we can easily obtain,

$$\det(\Phi_1 \mathbf{T}_{L_n} \Psi_1) = L_n \det \tilde{M}_{n-1}(\tilde{a}_1, [L_i]_{i=n-1}^2, [\tilde{b}_i]_{i=1}^{n-1}),$$

where

$$\tilde{M}_{n-1}(\tilde{a}_1, [L_i]_{i=n-1}^2, [\tilde{b}_i]_{i=1}^{n-1}) =$$

$$\begin{pmatrix}
\tilde{a}_1 & L_{n-1} & L_{n-2} & L_{n-3} & \cdots & L_4 & L_3 & L_2 \\
\tilde{b}_1 & \tilde{b}_2 & \tilde{b}_3 & \tilde{b}_4 & \cdots & \tilde{b}_{n-3} & \tilde{b}_{n-2} & \tilde{b}_{n-1} \\
0 & 2L_2 & L_1 & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & L_4 & 2L_2 & \ddots & \ddots & & & \vdots \\
\vdots & 2L_3 & L_4 & \ddots & \ddots & \ddots & & \vdots \\
\vdots & 2L_4 & 2L_3 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 2L_{n-3} & 2L_{n-4} & \cdots & \cdots & L_4 & 2L_2 & L_1
\end{pmatrix}_{(n-1)\times(n-1)}.$$

We apply methods of elementary row transformation to this matrix for computing the determinant of it, thus we can obtain $\tilde{M}_{n-1}^{'}(\tilde{a}_1, [L_i]_{i=n-1}^2, [\tilde{b}_i]_{i=1}^{n-1})$, it is of the form as,

$$\tilde{M}_{n-1}^{'}(\tilde{a}_1, [L_i]_{i=n-1}^2, [\tilde{b}_i]_{i=1}^{n-1}) =$$

$$\begin{pmatrix}
\tilde{a}_1 & L_{n-1} & L_{n-2} & L_{n-3} & \cdots & \cdots & \cdots & L_4 & L_3 & L_2 \\
\tilde{b}_1 & \tilde{b}_2 & \tilde{b}_3 & \tilde{b}_4 & \cdots & \cdots & \cdots & \tilde{b}_{n-3} & \tilde{b}_{n-2} & \tilde{b}_{n-1} \\
0 & 6 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & 7 & 6 & 1 & \ddots & & & & & \vdots \\
\vdots & -5 & 0 & 5 & 1 & \ddots & & & & \vdots \\
\vdots & -1 & -5 & 0 & 5 & 1 & \ddots & & & \vdots \\
\vdots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & \cdots & \cdots & 0 & -1 & -5 & 0 & 5 & 1
\end{pmatrix},$$

so we can obtain

$$\begin{aligned}
&\det \tilde{M}_{n-1}^{'}(\tilde{a}_1, [L_i]_{i=n-1}^2, [\tilde{b}_i]_{i=1}^{n-1}) = \\
&\qquad \tilde{a}_1 \det \tilde{\mathcal{C}}_{n-2}([\tilde{b}_i]_{i=2}^{n-1}, -1, -5, 0, 5, 1) - \tilde{b}_1 \det \tilde{\mathcal{D}}_{n-2}([L_i]_{i=n-1}^2, -1, -5, 0, 5, 1),
\end{aligned}$$

where $\tilde{\mathcal{C}}_{n-2}([\tilde{b}_i]_{i=2}^{n-1}, -1, -5, 0, 5, 1)$ and $\tilde{\mathcal{D}}_{n-2}([L_i]_{i=n-1}^2, -1, -5, 0, 5, 1)$ are the forms as in the interpretation of Theorem 3.1.

To obtain determinants of matrices $\tilde{\mathcal{C}}_{n-2}([\tilde{b}_i]_{i=2}^{n-1}, -1, -5, 0, 5, 1)$ and $\tilde{\mathcal{D}}_{n-2}([L_i]_{i=n-1}^{2}, -1, -5, 0, 5, 1)$, develop them in accordance with the last column, and in turn it, we can get recursive formulas (3.2)-(3.6). And observe that □

$$\det \Phi_1 = \det \Psi_1 = (-1)^{\frac{(n-1)(n-2)}{2}},$$

so we can get $\det \mathbf{T}_{L_n}$, which completes the proof.

Theorem 3.2. *Let $\mathbf{T}_{L_n}$ be a Lucas skew symmetric Toeplitz matrix as the form of (1.2). If $\mathbf{T}_{L_n}$ is a nonsingular matrix, then*

$$\mathbf{T}_{L_n}^{-1} = \begin{pmatrix} 0 & \tilde{\rho}_{12} & \tilde{\rho}_{13} & \tilde{\rho}_{14} & \cdots & \tilde{\rho}_{1,n-1} & \tilde{\rho}_{1n} \\ -\tilde{\rho}_{12} & 0 & \tilde{\rho}_{23} & \tilde{\rho}_{24} & \cdots & \tilde{\rho}_{2,n-1} & \tilde{\rho}_{1,n-1} \\ -\tilde{\rho}_{13} & -\tilde{\rho}_{23} & 0 & \tilde{\rho}_{34} & \cdots & \tilde{\rho}_{2,n-2} & \tilde{\rho}_{1,n-2} \\ -\tilde{\rho}_{14} & -\tilde{\rho}_{24} & -\tilde{\rho}_{34} & 0 & \cdots & \tilde{\rho}_{2,n-3} & \tilde{\rho}_{1,n-3} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ -\tilde{\rho}_{1,n-1} & -\tilde{\rho}_{2,n-1} & -\tilde{\rho}_{2,n-2} & -\tilde{\rho}_{2,n-3} & \cdots & 0 & \tilde{\rho}_{12} \\ -\tilde{\rho}_{1,n} & -\tilde{\rho}_{1,n-1} & -\tilde{\rho}_{1,n-2} & -\tilde{\rho}_{1,n-3} & \cdots & -\tilde{\rho}_{12} & 0 \end{pmatrix}, \tag{3.7}$$

that is to say $\mathbf{T}_{L_n}^{-1}$ is skew symmetric about diagonal and also symmetric about secondary diagonal, where

$$\tilde{\rho}_{12} = \tilde{\delta}_{1n}, \quad \tilde{\rho}_{13} = \tilde{\delta}_{1,n-1} + \tilde{\delta}_{1n}, \quad \tilde{\rho}_{23} = \tilde{\delta}_{2,n-1} + \tilde{\delta}_{2n}, \quad \tilde{\rho}_{1n} = \tilde{\delta}_{11} - \frac{L_{n-1}}{L_n}\tilde{\delta}_{13} - \tilde{\delta}_{14},$$

$$\tilde{\rho}_{mj} = \tilde{\delta}_{m,n+2-j} + \tilde{\delta}_{m,n+3-j} - \tilde{\delta}_{m,n+4-j}, \ (m = 1, 2, \cdots, n; j = 4, 5, \cdots, n-1),$$

$$\tilde{\delta}_{11} = \frac{1}{-L_n}, \quad \tilde{\delta}_{12} = \frac{\tilde{d}_1}{\tilde{a}_1}, \quad \tilde{\delta}_{13} = \frac{\tilde{d}_2}{\tilde{\ell}_1} + \sum_{i=1}^{n-3} \tilde{d}_{i+2}\tilde{\xi}_i^{'}, \quad \tilde{\delta}_{1j} = \tilde{d}_2\tilde{\eta}_{j-3}^{'} + \sum_{i=1}^{n-3} \tilde{d}_{i+2}\tilde{\sigma}_{i,j-3}, \ (j = 4, 5, \cdots, n),$$

$$\tilde{\delta}_{m1} = 0, \ (m = 2, 3, \cdots, n), \qquad \tilde{\delta}_{m2} = \frac{x_{n+1-m}}{\tilde{a}_1}, \ (m = 2, 3, \cdots, n),$$

$$\tilde{\delta}_{m3} = \frac{\tilde{e}_2 x_{n+1-m}}{\tilde{\ell}_1} + \sum_{i=1}^{n-3} \tilde{e}_{i+2} x_{n+1-m}\tilde{\xi}_i^{'} + \tilde{\xi}_{n-1-m}^{'}, \ (m = 2, 3, \cdots, n),$$

$$\tilde{\delta}_{mj} = \tilde{e}_2 x_{n+1-m}\tilde{\eta}_{j-3}^{'} + \sum_{i=1}^{n-3} \tilde{e}_{i+2} x_{n+1-m}\tilde{\sigma}_{i,j-3} + \tilde{\sigma}_{n-1-m,j-3}, \ (m = 2, 3, \cdots, n; \ j = 4, 5, \cdots, n),$$

$$\tilde{d}_1 = \frac{\tilde{a}_2}{L_n}, \quad \tilde{d}_i = \frac{-\tilde{a}_2 L_{n+1-i} - \tilde{a}_1 L_i}{\tilde{a}_1 L_n}, \ (i = 2, 3, \cdots, n-1), \quad \tilde{e}_i = \frac{-L_{n+1-i}}{\tilde{a}_1}, \ (i = 2, 3, \cdots, n-1),$$

$$\tilde{\ell}_1 = -\frac{\tilde{b}_1}{\tilde{a}_1} L_{n-1} + \tilde{b}_2 - \tilde{V}_1 \tilde{W}_1^{-1} \tilde{U}_1, \quad \tilde{U}_1 = (2L_2, L_4, 2L_3, \cdots, 2L_{n-4}, 2L_{n-3})^T$$

$$\tilde{V}_1 = (-\frac{\tilde{b}_1}{\tilde{a}_1} L_{n-2} + \tilde{b}_3, -\frac{\tilde{b}_1}{\tilde{a}_1} L_{n-3} + \tilde{b}_4, \cdots, -\frac{\tilde{b}_1}{\tilde{a}_1} L_3 + \tilde{b}_{n-2}, -\frac{\tilde{b}_1}{\tilde{a}_1} L_2 + \tilde{b}_{n-1}),$$

$$\tilde{W}_1^{-1} = (\tilde{a}_{i,j})_{i,j=1}^{n-3}, \quad \tilde{a}_{i,j} = \begin{cases} \tilde{\beta}_{i-j+1}, & i \geqslant j, \\ 0, & i < j, \end{cases} \quad (i, j = 1, 2, \cdots, n-3), \quad \tilde{\beta}_i = (-1)^{i-1}\tilde{p}_i,$$

$$\tilde{p}_1 = 1, \quad \tilde{p}_2 = 6, \quad \tilde{p}_i = 6\tilde{p}_{i-1} - 7\tilde{p}_{i-2} + \sum_{j=2}^{i-2} (-1)^j 2L_{j+1}\tilde{p}_{i-j-1}, \ (i = 3, 4, 5, \cdots, n-3),$$

$\tilde{\eta}_m^{'} = -\frac{1}{\tilde{\ell}_1}\tilde{\eta}_m,\ (m = 1, 2, \cdots, n-3),\quad \tilde{\xi}_m^{'} = -\frac{1}{\tilde{\ell}_1}\tilde{\xi}_m,\ (m = 1, 2, \cdots, n-3),$

$\tilde{\eta}_m = \sum_{i=m}^{n-3}(-\frac{\tilde{b}_1}{\tilde{a}_1}L_{n-1-i} + \tilde{b}_{i+2})\tilde{\beta}_{i+1-m},\ (m = 1, 2, \cdots, n-3),$

$\tilde{\xi}_1 = 6\tilde{\beta}_1,\quad \tilde{\xi}_2 = 6\tilde{\beta}_2 + 7\tilde{\beta}_1,\quad \tilde{\xi}_m = 6\tilde{\beta}_m + 7\tilde{\beta}_{m+1} + \sum_{i=1}^{m-2}\tilde{\beta}_{m+1-i}\cdot 2L_{i+2},\ (m = 3, 4, \cdots, n-3),$

$\tilde{\sigma}_{i,j} = \tilde{\beta}_{i-j+3} + \frac{1}{\tilde{\ell}_1}\tilde{\xi}_i\tilde{\eta}_j,\ (i, j = 1, 2, \cdots, n-3),$

$\tilde{a}_1,\ \tilde{a}_2,\ \tilde{b}_i,\ x_i,\ (i = 1, 2, \cdots, n-1)$ *are the same as in* Theorem 3.1.

Proof. Let Φ_2 and Ψ_2 be two $n \times n$ invertible matrices, defined by

$$\Phi_2 = \begin{pmatrix} 0 & 1 & & & & & \\ 1 & & & & & & \\ -\frac{\tilde{b}_1}{\tilde{a}_1} & 0 & 1 & & & & \\ & & & 1 & & & \\ & & & & \ddots & & \\ & & & & & 1 & \\ & & & & & & 1 \end{pmatrix},\quad \Psi_2 = \begin{pmatrix} 1 & \tilde{d}_1 & \tilde{d}_2 & \tilde{d}_3 & \cdots & \tilde{d}_{n-2} & \tilde{d}_{n-1} \\ & 1 & \tilde{e}_2 & \tilde{e}_3 & \cdots & \tilde{e}_{n-2} & \tilde{e}_{n-1} \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & & & \ddots & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix},$$

where $\tilde{d}_1 = \frac{\tilde{a}_2}{L_n},\ \tilde{d}_i = \frac{-\tilde{a}_2 L_{n+1-i} - \tilde{a}_1 L_i}{\tilde{a}_1 L_n},\ \tilde{e}_i = \frac{-L_{n+1-i}}{\tilde{a}_1},\ (i = 2, 3, \cdots, n-1).$

Let Φ_1 and Ψ_1 be as in the proof of Theorem 3.1, multiplying $\Phi_1\mathbf{T}_{L_n}\Psi_1$ by Φ_2 from the left and by Ψ_2 from the right, we obtain

$$\Phi_2\Phi_1\mathbf{T}_{L_n}\Psi_1\Psi_2$$
$$= \begin{pmatrix} -L_n & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \tilde{a}_1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & -\frac{\tilde{b}_1}{\tilde{a}_1}L_{n-1} + \tilde{b}_2 & -\frac{\tilde{b}_1}{\tilde{a}_1}L_{n-2} + \tilde{b}_3 & \cdots & \cdots & -\frac{\tilde{b}_1}{\tilde{a}_1}L_3 + \tilde{b}_{n-2} & -\frac{\tilde{b}_1}{\tilde{a}_1}L_2 + \tilde{b}_{n-1} \\ \vdots & \vdots & 2L_2 & L_1 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & L_4 & 2L_2 & \ddots & \ddots & & \vdots \\ \vdots & \vdots & 2L_3 & L_4 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 2L_{n-3} & 2L_{n-4} & \cdots & L_4 & 2L_2 & L_1 \end{pmatrix},$$

this matrix admits a block partition of the form as

$$\Phi_2\Phi_1\mathbf{T}_{L_n}\Psi_1\Psi_2 = \tilde{\mathbf{N}} \oplus \tilde{\mathbf{R}}\ ,$$

where $\tilde{\mathbf{N}} \oplus \tilde{\mathbf{R}}$ is the direct sum of $\tilde{\mathbf{N}}$ and $\tilde{\mathbf{R}}$, $\tilde{\mathbf{N}} = \mathrm{diag}(-L_n, \tilde{a}_1)$ is a nonsingular diagonal matrix, and

$$\tilde{\mathbf{R}} = \begin{pmatrix} -\frac{\tilde{b}_1}{\tilde{a}_1}L_{n-1}+\tilde{b}_2 & -\frac{\tilde{b}_1}{\tilde{a}_1}L_{n-2}+\tilde{b}_3 & -\frac{\tilde{b}_1}{\tilde{a}_1}L_{n-3}+\tilde{b}_4 & \cdots & -\frac{\tilde{b}_1}{\tilde{a}_1}L_3+\tilde{b}_{n-2} & -\frac{\tilde{b}_1}{\tilde{a}_1}L_2+\tilde{b}_{n-1} \\ 2L_2 & L_1 & 0 & \cdots & 0 & 0 \\ L_4 & 2L_2 & L_1 & \ddots & & \vdots \\ 2L_3 & L_4 & 2L_2 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 2L_{n-3} & 2L_{n-4} & 2L_{n-5} & \cdots & 2L_2 & L_1 \end{pmatrix}.$$

Let $\Phi = \Phi_2\Phi_1$ and $\Psi = \Psi_1\Psi_2$, we can get,

$$\mathbf{T}_{L_n}^{-1} = \Psi(\tilde{\mathbf{N}}^{-1} \oplus \tilde{\mathbf{R}}^{-1})\Phi,$$

where

$$\Phi = \begin{pmatrix} 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \\ 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ -\frac{\tilde{b}_1}{\tilde{a}_1} & \vdots & & & & \iddots & 1 & -\frac{L_{n-1}}{L_n} \\ 0 & \vdots & & & \iddots & 1 & 1 & -1 \\ \vdots & \vdots & & \iddots & \iddots & \iddots & \iddots & 0 \\ \vdots & \vdots & \iddots & \iddots & \iddots & \iddots & \iddots & \vdots \\ \vdots & 0 & 1 & 1 & -1 & \iddots & & \vdots \\ 0 & 1 & 1 & -1 & 0 & \cdots & \cdots & 0 \end{pmatrix},$$

and

$$\Psi = \begin{pmatrix} 1 & \tilde{d}_1 & \tilde{d}_2 & \tilde{d}_3 & \cdots & \tilde{d}_{n-3} & \tilde{d}_{n-2} & \tilde{d}_{n-1} \\ 0 & x_{n-1} & \tilde{e}_2x_{n-1} & \tilde{e}_3x_{n-1} & \cdots & \tilde{e}_{n-3}x_{n-1} & \tilde{e}_{n-2}x_{n-1} & \tilde{e}_{n-1}x_{n-1}+1 \\ 0 & x_{n-2} & \tilde{e}_2x_{n-2} & \tilde{e}_3x_{n-2} & \cdots & \tilde{e}_{n-3}x_{n-2} & \tilde{e}_{n-2}x_{n-2}+1 & \tilde{e}_{n-1}x_{n-2} \\ 0 & x_{n-3} & \tilde{e}_2x_{n-3} & \tilde{e}_3x_{n-3} & \cdots & \tilde{e}_{n-3}x_{n-3}+1 & \tilde{e}_{n-2}x_{n-3} & \tilde{e}_{n-1}x_{n-3} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & x_3 & \tilde{e}_2x_3 & \tilde{e}_3x_3+1 & \cdots & \tilde{e}_{n-3}x_3 & \tilde{e}_{n-2}x_3 & \tilde{e}_{n-1}x_3 \\ 0 & x_2 & \tilde{e}_2x_2+1 & \tilde{e}_3x_2 & \cdots & \tilde{e}_{n-3}x_2 & \tilde{e}_{n-2}x_2 & \tilde{e}_{n-1}x_2 \\ 0 & x_1 & \tilde{e}_2x_1 & \tilde{e}_3x_1 & \cdots & \tilde{e}_{n-3}x_1 & \tilde{e}_{n-2}x_1 & \tilde{e}_{n-1}x_1 \end{pmatrix}.$$

Observe that the inverse matrix of $\tilde{\mathbf{N}}$ is of the form as,

$$\tilde{\mathbf{N}}^{-1} = \mathrm{diag}(-L_n^{-1},\ \tilde{a}_1^{-1}).$$

Let $\tilde{\mathbf{R}} = \begin{pmatrix} -\frac{\tilde{b}_1}{\tilde{a}_1}L_{n-1}+\tilde{b}_2 & \tilde{V}_1 \\ \tilde{U}_1 & \tilde{W}_1 \end{pmatrix}$ be an $(n-2)\times(n-2)$ matrix, where $\tilde{U}_1 = (2L_2, L_4, 2L_3, \cdots,$

$2L_{n-4}, 2L_{n-3})^T$, $\tilde{V}_1 = (-\frac{\tilde{b}_1}{\tilde{a}_1}L_{n-2} + \tilde{b}_3, -\frac{\tilde{b}_1}{\tilde{a}_1}L_{n-3} + \tilde{b}_4, \cdots, -\frac{\tilde{b}_1}{\tilde{a}_1}L_2 + \tilde{b}_{n-1})$,

$$\tilde{W}_1 = \begin{pmatrix} L_1 & & & & & & \\ 2L_2 & L_1 & & & & & \\ L_4 & 2L_2 & L_1 & & & & \\ 2L_3 & \ddots & \ddots & \ddots & & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & & \\ 2L_{n-5} & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 2L_{n-4} & 2L_{n-5} & \cdots & 2L_3 & L_4 & 2L_2 & L_1 \end{pmatrix},$$

as $L_1 = 1$, so $\tilde{W}_1$ is an invertible matrix. Use Lemma 1.1 we can obtain the inverse matrix of $\tilde{W}_1$, $\tilde{W}_1^{-1} = (\tilde{a}_{i,j})_{i,j=1}^{n-3}$, as in the interpretation of Theorem 3.2.

Use Lemma 5 in [23], we obtain,

$$\tilde{\mathbf{R}}^{-1} = \begin{pmatrix} \frac{1}{\tilde{\ell}_1} & -\frac{1}{\tilde{\ell}_1}\tilde{V}_1\tilde{W}_1^{-1} \\ -\frac{1}{\tilde{\ell}_1}\tilde{W}_1^{-1}\tilde{U}_1 & \tilde{W}_1^{-1} + \frac{1}{\tilde{\ell}_1}\tilde{W}_1^{-1}\tilde{U}_1\tilde{V}_1\tilde{W}_1^{-1} \end{pmatrix},$$

where $\tilde{\ell}_1 = -\frac{\tilde{b}_1}{\tilde{a}_1}L_{n-1} + \tilde{b}_2 - \tilde{V}_1\tilde{W}_1^{-1}\tilde{U}_1$, and simply we can get $(\tilde{\mathbf{N}} \oplus \tilde{\mathbf{R}})^{-1}$,

$$(\tilde{\mathbf{N}} \oplus \tilde{\mathbf{R}})^{-1} = \begin{pmatrix} \frac{1}{-L_n} & 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & \frac{1}{\tilde{a}_1} & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \frac{1}{\tilde{\ell}_1} & \tilde{\eta}_1^{'} & \tilde{\eta}_2^{'} & \cdots & \cdots & \tilde{\eta}_{n-4}^{'} & \tilde{\eta}_{n-3}^{'} \\ 0 & 0 & \tilde{\xi}_1^{'} & \tilde{\sigma}_{11} & \tilde{\sigma}_{12} & \cdots & \cdots & \tilde{\sigma}_{1,n-4} & \tilde{\sigma}_{1,n-3} \\ 0 & 0 & \tilde{\xi}_2^{'} & \tilde{\sigma}_{21} & \tilde{\sigma}_{22} & \cdots & \cdots & \tilde{\sigma}_{2,n-4} & \tilde{\sigma}_{2,n-3} \\ 0 & 0 & \tilde{\xi}_3^{'} & \tilde{\sigma}_{31} & \tilde{\sigma}_{32} & \cdots & \cdots & \tilde{\sigma}_{3,n-4} & \tilde{\sigma}_{3,n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & & \vdots & \vdots \\ 0 & 0 & \tilde{\xi}_{n-4}^{'} & \tilde{\sigma}_{n-4,1} & \tilde{\sigma}_{n-4,2} & \cdots & \cdots & \tilde{\sigma}_{n-4,n-4} & \tilde{\sigma}_{n-4,n-3} \\ 0 & 0 & \tilde{\xi}_{n-3}^{'} & \tilde{\sigma}_{n-3,1} & \tilde{\sigma}_{n-3,2} & \cdots & \cdots & \tilde{\sigma}_{n-3,n-4} & \tilde{\sigma}_{n-3,n-3} \end{pmatrix},$$

then multiplying $(\tilde{\mathbf{N}} \oplus \tilde{\mathbf{R}})^{-1}$ by Ψ from the left, we can get,

$$\Psi(\tilde{\mathbf{N}} \oplus \tilde{\mathbf{R}})^{-1} = \begin{pmatrix} \tilde{\delta}_{11} & \tilde{\delta}_{12} & \tilde{\delta}_{13} & \cdots & \tilde{\delta}_{1,n-1} & \tilde{\delta}_{1,n} \\ \tilde{\delta}_{21} & \tilde{\delta}_{22} & \tilde{\delta}_{23} & \cdots & \tilde{\delta}_{2,n-1} & \tilde{\delta}_{2,n} \\ \tilde{\delta}_{31} & \tilde{\delta}_{32} & \tilde{\delta}_{33} & \cdots & \tilde{\delta}_{3,n-1} & \tilde{\delta}_{3,n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \tilde{\delta}_{n-1,1} & \tilde{\delta}_{n-1,2} & \tilde{\delta}_{n-1,3} & \cdots & \tilde{\delta}_{n-1,n-1} & \tilde{\delta}_{n-1,n} \\ \tilde{\delta}_{n,1} & \tilde{\delta}_{n,2} & \tilde{\delta}_{n,3} & \cdots & \tilde{\delta}_{n,n-1} & \tilde{\delta}_{n,n} \end{pmatrix}.$$

In the end, we can obtain $\mathbf{T}_{L_n}^{-1}$,

$$\mathbf{T}_{L_n}^{-1} = \Psi(\tilde{\mathbf{N}} \oplus \tilde{\mathbf{R}})^{-1}\Phi = \begin{pmatrix} 0 & \tilde{\rho}_{12} & \tilde{\rho}_{13} & \tilde{\rho}_{14} & \cdots & \tilde{\rho}_{1,n-1} & \tilde{\rho}_{1n} \\ -\tilde{\rho}_{12} & 0 & \tilde{\rho}_{23} & \tilde{\rho}_{24} & \cdots & \tilde{\rho}_{2,n-1} & \tilde{\rho}_{1,n-1} \\ -\tilde{\rho}_{13} & -\tilde{\rho}_{23} & 0 & \tilde{\rho}_{3,4} & \cdots & \tilde{\rho}_{2,n-2} & \tilde{\rho}_{1,n-2} \\ -\tilde{\rho}_{14} & -\tilde{\rho}_{24} & -\tilde{\rho}_{3,4} & 0 & \cdots & \tilde{\rho}_{2,n-3} & \tilde{\rho}_{1,n-3} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ -\tilde{\rho}_{1,n-1} & -\tilde{\rho}_{2,n-1} & -\tilde{\rho}_{2,n-2} & -\tilde{\rho}_{2,n-3} & \cdots & 0 & \tilde{\rho}_{12} \\ -\tilde{\rho}_{1n} & -\tilde{\rho}_{1,n-1} & -\tilde{\rho}_{1,n-2} & -\tilde{\rho}_{1,n-3} & \cdots & -\tilde{\rho}_{12} & 0 \end{pmatrix},$$

where

$$\tilde{\rho}_{12} = \tilde{\delta}_{1n}, \quad \tilde{\rho}_{13} = \tilde{\delta}_{1,n-1} + \tilde{\delta}_{1n}, \quad \tilde{\rho}_{23} = \tilde{\delta}_{2,n-1} + \tilde{\delta}_{2n}, \quad \tilde{\rho}_{1n} = \tilde{\delta}_{11} - \frac{L_{n-1}}{L_n}\tilde{\delta}_{13} - \tilde{\delta}_{14},$$

$$\tilde{\rho}_{mj} = \tilde{\delta}_{m,n+2-j} + \tilde{\delta}_{m,n+3-j} - \tilde{\delta}_{m,n+4-j}, \ (m = 1, 2, \cdots, n; j = 4, 5, \cdots, n-1),$$

which completes the proof. □

4 Numerical Example

In this section, an example demonstrates the method introduced above for the calculation of determinant and inverse of the Fibonacci skew symmetric Toeplitz matrix. Here we consider a 6×6 matrix:

$$\mathbf{T}_{F_6} = \begin{pmatrix} 0 & 1 & 2 & 3 & 5 & 8 \\ -1 & 0 & 1 & 2 & 3 & 5 \\ -2 & -1 & 0 & 1 & 2 & 3 \\ -3 & -2 & -1 & 0 & 1 & 2 \\ -5 & -3 & -2 & -1 & 0 & 1 \\ -8 & -5 & -3 & -2 & -1 & 0 \end{pmatrix}_{6\times 6}.$$

Use the corresponding formulas in Theorem 2.1, we get $a_1 = 1+\sqrt{2}i$, $b_1 = -\frac{1}{8}$ and from (2.2), (2.3) we can obtain

$$\det \mathcal{C}_4([b_i]_{i=2}^5, -1, -1, 0, 1, 1) = 0, \quad \det \mathcal{D}_4([F_i]_{i=5}^2, -1, -1, 0, 1, 1) = 1,$$

from (2.1), we obtain

$$\begin{aligned} \det \mathbf{T}_{F_6} &= F_6[a_1 \det \mathcal{C}_4([b_i]_{i=2}^5, -1, -1, 0, 1, 1) - b_1 \det \mathcal{D}_4([F_i]_{i=5}^2, -1, -1, 0, 1, 1)] \\ &= 1. \end{aligned}$$

As the inverse calculation, use the corresponding formulas in Theorem 2.2, we get $\rho_{12} = 0$, $\rho_{13} = -1$, $\rho_{14} = 2$, $\rho_{15} = -1$, $\rho_{16} = 0$, $\rho_{23} = 1$, $\rho_{24} = -3$, $\rho_{25} = 3$, $\rho_{34} = 0$, so we can get

$$\mathbf{T}_{F_6}^{-1} = \begin{pmatrix} 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -3 & 3 & -1 \\ 1 & -1 & 0 & 0 & -3 & 2 \\ -2 & 3 & 0 & 0 & 1 & -1 \\ 1 & -3 & 3 & -1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \end{pmatrix}.$$

5 Conclusion

In this paper, by constructing the special transformation matrices we give the determinant and inverse of the Fibonacci skew symmetric Toeplitz matrix in section 2, and give the determinant and inverse of the Lucas skew symmetric Toeplitz matrix in section 3.

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Competing Interest

Authors have declared that no competing interests exist.

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