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Homomorphism of intuitionistic fuzzy multigroups

I. M. Adamu

Department of Mathematics, Federal University Dutse, P.M.B. 7156, Dutse, Jigawa State, Nigeria.; idreesmuhammadadam@gmail.com

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Abstract: This paper introduces the concept of homomorphism in intuitionistic fuzzy multigroups context. It also investigates Some homomorphic properties of intuitionistic fuzzy multigroups. It is shown that the homomorphic image and homomorphic preimage of intuitionistic fuzzy multigroups are also intuitionistic fuzzy multigroups. Finally, it presents some homomorphic properties of normalizer of intuitionistic fuzzy multigroups.

Keywords: Intuitionistic fuzzy multiset, intuitionistic fuzzy multigroups, Homomorphism of intuitionistic fuzzy multigroups.

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1. Introduction

In modern mathematics, a set is a well-defined collection of distinct objects. Set theory was introduced by German mathematician George Ferdinand Ludwig Cantor(1845-1918). In classical sense, all mathematical notions including sets must be exact. However, if repeated occurrences of any object are allowed in a set, then the mathematical structure is called multiset [1]. Thus, a multiset differs from a set in the sense that each element has a multiplicity. An account of the development of multiset theory can be seen in [2–5]. Most of the real life situations are complex and modelling them we need a simplification of the complex system. The simplification must be in such a way that the information lost should be minimum. One way to do this is to allow some degree of uncertainty into it. To handle situations like this, Zadeh [6] proposed fuzzy sets. A fuzzy set has a membership function assigns to each element of the universe of discourse, a number from the unit interval [0,1] to indicate the degree of belongingness to the set under consideration. Fuzzy sets were introduced with a view to reconcile mathematical modelling and human knowledge in the engineering sciences. The theory of fuzzy sets has been applied to group theoretic notions [7–10].

Atanassov [11,12] introduced a generalized fuzzy sets called intuitionistic fuzzy set. In the same time, a theory called intuitionistic fuzzy set theory was independently introduced by Takeuti and Titani [13] as a theory developed in (a kind of) intuitionistic logic. Intuitionistic Fuzzy sets provide a flexible framework to explain uncertainty and vagueness. The theory of intuitionistic fuzzy sets has been applied to group theoretic notions [14–16]. As a generalization of multiset, Yager [17] introduced fuzzy multisets and suggested possible applications to relational databases. Shinoj *et al.,* [18] has studied the structure of groups in fuzzy multisets. Several researches on fuzzy multigroup theory have been conducted as seen in [19–25]. The concept of intuitionistic fuzzy multiset was proposed in [26] as a study of intuiionistic fuzzy sets in multiset framework. Some works have been done on both the theory and applications of intuitionistic fuzzy multisets [27–34]. In a way to apply intuitionistic fuzzy multisets to group theory, Shinoj and John [35] proposed intuitionistic fuzzy multigroups. Adamu *et al.,* [36] developed the concept of normal sub-intuitionistic fuzzy multigroups and investigate some of its related algebraic structures.

The motivation of this work is to establish the idea of homomorphism in intuitionistic fuzzy multigroups. This paper introduces the concept of homomorphism in intuitionistic fuzzy multigroups context and investigated some of its properties. The outline are presented as follows: Section 2 presents some foundational concepts relevant to the study whereas the main results are reported in Section 3. Section 4 summarises and concludes the paper.

2. Preliminaries

In this section we presents some existing definitions and results to be used in the sequel. Throughout the work IFMS(X) denotes the set of all intuitionistic fuzzy multisets of X, and IFMG(X) denote the set of all intuitionistic fuzzy multigroups of X, where X is a non-empty set.

Definition 1. [26] Let X be a nonempty set. An intuitionistic fuzzy multiset A of X is characterized by two count membership function CM_A and count non membership function CN_A defined by

$$CM_A\colon X \to Q \,\text{and}\, CN_A\colon X \to Q,$$

where Q is the set of all crisp multisets drawn from the unit interval $[0,1]$ such that for each $x \in X$, the membership sequence is defined as a decreasingly ordered sequence of elements in A which is denoted by $\mu_A^1(x), \mu_A^2(x), ..., \mu_A^p(x)$, where $\mu_A^1(x) \geq \mu_A^2(x) \geq, ..., \geq \mu_A^p(x)$ and the corresponding non membership sequence of elements in A is denoted by $(\nu_A^1(x), \nu_A^2(x), ..., \nu_A^p(x))$ such that $0 \leq \mu_A^i(x) + nu_A^i(x) \leq 1$ for every $\in X$ and $i = 1, 2, ..., p$.

An IFMS A is denoted by

$$A = \{(\mu_A^1(x), \mu_A^2(x), ..., \mu_A^p(x)), (\nu_A^1(x), \nu_A^2(x), ..., \nu_A^p(x)) >: x \in X\}.$$

Definition 2. [26] Length of an element x in an IFMS A is defined as the cardinality of $CM_A(x)$ or $CN_A(x)$ for which $0 \leq \mu_A^i(x) + nu_A^i(x) \leq 1$ and is denoted by $L(x : A)$. That is $L(x : A) = |CM_A(x)| = |CN_A(x)|$. If A and B are IFMSs drawn from X, then $L(x : A, B) = \max[L(x : A), L(x : B)]$. Alternatively we use $L(x)$ for $L(x : A, B)$.

Definition 3. [26] For any two IFMSs A and B of a set X, the following operations and relations hold.

(i) Inclusion: $A \subset B \Longleftrightarrow \mu_A^j(x) \leq \mu_B^j(x)$ and $\nu_A^j(x) \geq \nu_B^j(x)$, for $j = 1, 2, ..., L(x), \forall x \in X$.

(ii) Complement:
$$\overline{A} = \{(\nu_A^1(x), \nu_A^2(x), ..., \nu_A^p(x)), (\mu_A^1(x), \mu_A^2(x), ..., \mu_A^p(x)) >: x \in X\}.$$

(iii) Union: In $A \cup B$, the membership and non-membership values are obtained as follows:

$$\mu_{A\cup B}^j(x) = \mu_A^j(x) \vee \mu_B^j(x) \text{ and } \nu_{A\cup B}^j(x) = \nu_A^j(x) \wedge \nu_B^j(x),$$

for $j = 1, 2, ..., L(x), \forall x \in X$.

(iv) Intersection: In $A \cap B$, the membership and non-membership values are obtained as follows:

$$\mu_{A\cap B}^j(x) = \mu_A^j(x) \wedge \mu_B^j(x) \text{ and } \nu_{A\cap B}^j(x) = \nu_A^j(x) \vee \nu_B^j(x),$$

for $j = 1, 2, ..., L(x), \forall x \in X$.

Definition 4. [28] Let X and Y be two non-empty sets and $f\colon X \to Y$ be a mapping. Then

(i) the image of $A \in FMS(X)$ under the mapping f is an IFMS of Y denoted by $f(A)$, where

$$CM_{f(A)}(y) = \begin{cases} \bigvee_{f(x)=y} CM_A(x), & f^{-1}(y) \neq \emptyset \\ 0, & \textit{otherwise}. \end{cases}$$

Similarly,

$$CN_{f(A)}(y) = \begin{cases} \bigwedge_{f(x)=y} CN_A(x), & f^{-1}(y) \neq \emptyset \\ 0, & \textit{otherwise}. \end{cases}$$

(ii) the inverse image of $B \in FM(Y)$ under the mapping f is an IFMS of X denoted by $f^{-1}(B)$, where $CM_{f^{-1}(B)}(x) = CM_B(f(x))$ and $CN_{f^{-1}(B)}(x) = CN_B(f(x))$.

Definition 5. [35] Let X be a group. An intuitionistic fuzzy multiset G of X is an intuitionistic fuzzy multigroup (IFMG) of X if the counts (count membership and non-membership) of G satisfies the following two conditions:

(i) $CM_G(xy) \geq CM_G(x) \wedge CM_G(y)\ \forall x, y \in X$ and $CN_G(xy) \leq CN_G(x) \vee CN_G(y)\ \forall x, y \in X$,
(ii) $CM_G(x^{-1}) \geq CM_G(x)\ \forall x \in X$ and $CN_G(x^{-1}) \leq CN_G(x)\ \forall x \in X$.

Definition 6. [35] For any intuitionistic fuzzy multigroup $A \in IFMG(X)$,
$\exists$ its inverse, A^{-1}, defined by

$$CM_{A^{-1}}(x) = CM_A(x^{-1}) \forall x \in X \text{ and } CN_{A^{-1}}(x) = CN_A(x^{-1}) \forall x \in X.$$

Certainly, $A \in IFMG(X)$ if and only if $A^{-1} \in IFMG(X)$.

3. Main results

In this section, we introduce homomorphism of IFMSs and characterize it properties with some results.

Definition 7. Let X, Y be two groups and let $f \colon X \to Y$ be an isomorphism of groups. Suppose A and B are intuitionistic fuzzy multigroups of X and Y, respectively. Then, f induces a homomorphism from A to B which satisfies

(i) $CM_A(f^{-1}(y_1y_2)) \geq CM_A(f^{-1}(y_1)) \wedge CM_A(f^{-1}(y_2))$
and $CN_A(f^{-1}(y_1y_2)) \leq CN_A(f^{-1}(y_1)) \vee CN_A(f^{-1}(y_2))\ \forall y_1, y_2 \in Y$,
(ii) $CM_B(f(x_1x_2)) \geq CM_B(f(x_1)) \wedge CM_B(f(x_2))$
and $CN_B(f(x_1x_2)) \leq CN_B(f(x_1)) \vee CN_B(f(x_2))\ \forall x_1, x_2 \in X$,

where $f(A)$ and $f^{-1}(B)$ are as in Definition 4.

Definition 8. Let X and Y be groups and let $A \in IFMG(X)$ and $B \in IFMG(Y)$, respectively.

(i) A homomorphism f of X onto Y is called a weak homomorphism of A into B if $f(A) \subseteq B$. If f is a weak homomorphism of A into B, then we say that, A is weakly homomorphic to B denoted by $A \sim B$.
(ii) An isomorphism f of X onto Y is called a weak isomorphism of A into B if $f(A) \subseteq B$. If f is a weak isomorphism of A into B, then we say that, A is weakly isomorphic to B denoted by $A \simeq B$.
(iii) A homomorphism f of X onto Y is called a homomorphism of A onto B if $f(A) = B$. If f is a homomorphism of A onto B, then A is homomorphic to B denoted by $A \approx B$.
(iv) An isomorphism f of X onto Y is called an isomorphism of A onto B if $f(A) = B$. If f is an isomorphism of A onto B, then A is isomorphic to B denoted by $A \approxeq B$.

Remark 1. Let X and Y be groups and let $A \in IFMG(X)$ and $B \in IFMG(Y)$, respectively. Then

(i) a homomorphism f of X onto Y is called an epimorphism of A onto B if f is surjective.
(ii) a homomorphism f of X onto Y is called a monomorphism of A into B if f is injective.
(iii) a homomorphism f of X onto Y is called an endomorphism of A onto A if f is a map to itself.
(iv) a homomorphism f of X onto Y is called an automorphism of A onto A if f is both injective and surjective, that is, bijective.
(v) a homomorphism f of X onto Y is called an isomorphism of A onto B if f is both injective and surjective, that is, bijective.

Definition 9. Let A be a intuitionistic fuzzy submultigroup of $B \in IFMG(X)$. Then, the normalizer of A in B is given by

$$N(A) = \{g \in X \mid CM_A(gy) = CM_A(yg), CN_A(gy) = CN_A(yg)\ \forall y \in X\}.$$

Proposition 1. *Let $f \colon X \to Y$ be a homomorphism. For $A, B \in IFMG(X)$, if $A \subseteq B$, then $f(A) \subseteq f(B)$.*

Proof. Let $A, B \in IFMG(X)$ and $f : X \to Y$. Suppose $CM_A(x) \leq CM_B(x)$ and $CN_A(x) \leq CN_B(x)\ \forall\ x \in X$. Then it follows that

$$CM_{f(A)}(y) = CM_A(f^{-1}(y)) \leq CM_B(f^{-1}(y)) = CM_{f(B)}(y),$$

and

$$CN_{f(A)}(y) = CN_A(f^{-1}(y)) \leq CN_B(f^{-1}(y)) = CN_{f(B)}(y)\ \forall\ y \in Y.$$

Hence $f(A) \subseteq f(B)$. $\square$

Proposition 2. *Let X, Y be two groups and f be a homomorphism of X into Y for A, B $IFMG(Y)$, if $A \subseteq B$, then $f^{-1}(A) \subseteq f^{-1}(B)$.*

Proof. Given that $A, B \in IFMG(X)$ and $f : X \to Y$. Suppose $CM_A(y) \leq CM_B(y)$ and $CN_A(y) \leq CN_B(y)\ \forall\, y \in Y$. Then we have

$$CM_{f^{-1}(A)}(x) = CM_A(f(x)) \leq CM_B(f(x)) = CM_{f^{-1}(B)}(x),$$

Similarly,

$$CN_{f^{-1}(A)}(x) = CN_A(f(x)) \leq CN_B(f(x)) = CN_{f^{-1}(B)}(x) \forall x \in X.$$

□

Definition 10. Let f be a homomorphism of a group X into a group Y , and $A \in IFMG(X)$. If for all $x, y \in X$, $f(x) = f(y)$ implies $CM_A(x) = CM_A(y)$ and $CN_A(x) = CN_A(y)$ then, A is $f-$invariant.

Lemma 1. *Let $f : X \to Y$ be groups homomorphism and $A \in IFMG(X)$. If $\forall x \in X, f(x) = f(y)$, then, A is $f-$invariant.*

Proof. Suppose $f(x) = f(y) \forall x \in X$. Then,

$$CM_{f(A)}(f(x)) = CM_{f(A)}(f(y)) \text{ and } CN_{f(A)}(f(x)) = CN_{f(A)}(f(y)).$$

This implies $CM_A(x) = CM_A(y)$ and $CN_A(x) = CN_A(y)$. Hence, A is $f-$invariant. □

Lemma 2. *If $f : X \to Y$ is a homomorphism and $A \in IFMG(X)$. Then*

(i) $f(A^{-1}) = (f(A))^{-1}$.
(ii) $f^{-1}(f(A^{-1})) = f((f(A))^{-1})$.

Proof. (i) Let $y \in Y$. Then, we get

$$\begin{aligned} CM_{f(A^{-1})}(y) &= CM_{A^{-1}}(f^{-1}(y)) = CM_A(f^{-1}(y)) \\ &= CM_{f(A)}(y) = CM_{(f(A))^{-1}}(y) \forall y \in Y. \end{aligned}$$

Similarly,

$$\begin{aligned} CN_{f(A^{-1})}(y) &= CN_{A^{-1}}(f^{-1}(y)) = CN_A(f^{-1}(y)) \\ &= CN_{f(A)}(y) = CN_{(f(A))^{-1}}(y) \forall y \in Y. \end{aligned}$$

Hence $f(A^{-1}) = (f(A))^{-1}$.

(ii) Let $y \in Y$. Then, we get

$$\begin{aligned} CM_{f^{-1}(f(A^{-1}))}(y) &= CM_{f(A^{-1})}(f(y)) = CM_{A^{-1}}(f((f^{-1}(y))) \\ &= CM_A f((f^{-1}(y)) = CM_{f^{-1}(f(A))}(y) \\ &= CM_{f((f(A))^{-1})}(y) \forall y \in Y. \end{aligned}$$

Similarly,

$$\begin{aligned} CN_{f^{-1}(f(A^{-1}))}(y) &= CN_{f(A^{-1})}(f(y)) = CN_{A^{-1}}(f((f^{-1}(y))) \\ &= CN_A f((f^{-1}(y)) = CN_{f^{-1}(f(A))}(y) \\ &= CN_{f((f(A))^{-1})}(y) \forall y \in Y. \end{aligned}$$

Hence $f^{-1}(f(A^{-1})) = f((f(A))^{-1})$.
□

Proposition 3. *Let X and Y be groups such that $f : X \to Y$ is an isomorphic mapping. If $A \in IFMG(X)$ and $B \in IFMG(Y)$. Then*

(i) $(f^{-1}(B))^{-1} = f^{-1}(B^{-1})$.
(ii) $f^{-1}(f(A)) = f^{-1}(f(f^{-1}(B)))$.

Proof. Recall that if f is an isomorphism, then $f(x) = y \; \forall y \in Y$, consequently, $f(A) = B$.

(i)

$$\begin{aligned} CM_{(f^{-1}(B))^{-1}}(x) &= CM_{f^{-1}(B)}(x^{-1}) = CM_{f^{-1}(B)}(x) \\ &= CM_B(f(x)) = CM_{B^{-1}}((f(x))^{-1}) \\ &= CM_{B^{-1}}(f(x)) = CM_{f^{-1}(B^{-1})}(x). \end{aligned}$$

Similarly,

$$\begin{aligned} CN_{(f^{-1}(B))^{-1}}(x) &= CN_{f^{-1}(B)}(x^{-1}) = CN_{f^{-1}(B)}(x) \\ &= CN_B(f(x)) = CN_{B^{-1}}((f(x))^{-1}) \\ &= CN_{B^{-1}}(f(x)) = CN_{f^{-1}(B^{-1})}(x). \end{aligned}$$

Hence, $(f^{-1}(B))^{-1} = f^{-}(B^{-1})$.

(ii) Similar to (i).

□

Proposition 4. *Let $f : X \to Y$ be a homomorphism of groups. If $\{A_i\}_{i\in I} \in IFMG(X) and \{B_i\}_{i\in I} \in IFMG(Y)$ respectively. Then*

(i) $f(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} f(A_i)$.
(ii) $f(\bigcap_{i\in I} A_i) = \bigcap_{i\in I} f(A_i)$.
(iii) $f^{-1}(\bigcap_{i\in I} B_i) = \bigcap_{i\in I} f^{-1}(B_i)$
(iv) $f^{-1}(\bigcup_{i\in I} B_i) = \bigcup_{i\in I} f^{-1}(B_i)$

Proof. (i) Let $x \in X$ and $y \in Y$. Since f is a homomorphism, so $f(x) = y$. Then, we have

$$\begin{aligned} CM_{f(\bigcup_{i\in I} A_i)}(y) &= CM_{\bigcup_{i\in I} A_i}(f^{-1}(y)) \\ &= \bigvee_{i\in I} CM_{A_i}(f^{-1}(y)) \\ &= \bigvee_{i\in I} CM_{f(A_i)}(y) \\ &= CM_{\bigcup_{i\in I} f(A_i)}(y) \; \forall y \in Y \end{aligned}$$

Similarly,

$$\begin{aligned} CN_{f(\bigcap_{i\in I} A_i)}(y) &= CN_{\bigcap_{i\in I} A_i}(f^{-1}(y)) \\ &= \bigwedge_{i\in I} CN_{A_i}(f^{-1}(y)) \\ &= \bigwedge_{i\in I} CN_{f(A_i)}(y) \\ &= CN_{\bigcap_{i\in I} f(A_i)}(y) \; \forall y \in Y. \end{aligned}$$

Hence, $f(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} f(A_i)$.

The proofs of (ii)-(iv) are similar to (i).

□

Proposition 5. *Let X be group and $f : X \to Y$ be an automorphism. If $A \in IFMG(X)$, then $f(A) = A$ if and only if $f^{-1}(A) = A$. Consequently, $f(A) = f^{-1}(A)$.*

Proof. Let $f(x) = x\ \forall x \in X$ since f is an automorphism. Suppose $f(A) = A$, we have

$$\begin{aligned} CM_{f(A)}(x) &= CM_A(f^{-1}(x)) = CM_A(x) \\ &= CM_A(f^{-1}(x)) = CM_{f(A)}(x). \end{aligned}$$

Similarly,

$$\begin{aligned} CN_{f(A)}(x) &= CN_A(f^{-1}(x)) = CN_A(x) \\ &= CN_A(f^{-1}(x)) = CN_{f(A)}(x). \end{aligned}$$

Thus $f^{-1}(A) = A$.
Conversely, let $f^{-1}(A) = A$, we have

$$\begin{aligned} CM_{f^{-1}(A)}(x) &= CM_A(f(x)) = CM_A(x) \\ &= CM_A(f^{-1}(x)) = CM_{f(A)}(x). \end{aligned}$$

Similarly,

$$\begin{aligned} CN_{f^{-1}(A)}(x) &= CN_A(f(x)) = CN_A(x) \\ &= CN_A(f^{-1}(x)) = CN_{f(A)}(x). \end{aligned}$$

Thus $f(A) = A$. Hence $f(A) = A \Leftrightarrow f^{-1}(A) = A$. □

Proposition 6. *Let $f\colon X \to Y$ be a homomorphism. If $A \in IFMG(X)$, then $f^{-1}(f(A)) = A$ whenever f is injective.*

Proof. Suppose f is injective, then $f(x) = y\ \forall x \in X$ and $\forall y \in Y$. Now

$$\begin{aligned} CM_{f^{-1}(f(A))}(x) &= CM_{f(A)}(f(x)) = CM_{f(A)}(y) \\ &= CM_A(f^{-1}(y)) = CM_A(x). \end{aligned}$$

Also,

$$\begin{aligned} CN_{f^{-1}(f(A))}(x) &= CN_{f(A)}(f(x)) = CN_{f(A)}(y) \\ &= CN_A(f^{-1}(y)) = CN_A(x). \end{aligned}$$

Hence, $f^{-1}(f(A)) = A$. □

Corollary 1. *Let $f\colon X \to Y$ be a homomorphism. If $B \in IFMG(Y)$, then $f(f^{-1}(B)) = B$ whenever f is surjective.*

Proof. Similar to Proposition 6. □

Proposition 7. *Let X, Y and Z be groups and $f\colon X \to Y$ and $f\colon Y \to Z$ be homomorphisms. If $\{A_i\}_{i\in I} \in IFMG(X)$ and $\{B_i\}_{i\in I} \in IFMG(Y)$ and $i \in I$. Then*

(i) $f(A_i) \subseteq B_i \implies A_i \subseteq f^{-1}(B_i)$.
(ii) $g[f(A_i)] = [gf](A_i)$.
(iii) $f^{-1}[g^{-1}(B_i)] = [gf]^{-1}(Bi)$.

Proof. (i) The proof of (i) is trivial.
(ii) Since f and g are homomorphisms, then, $f(x) = y$ and $g(y) = z\ \forall x \in X$, $\forall y \in Y$ and $\forall z \in Z$, respectively. Now

$$\begin{aligned} CM_{g[f(A_i)]}(z) &= CM_{f(A_i)}(g^{-1}(z)) = CM_{f(A_i)}(y) \\ &= CM_{A_i}(f^{-1}(y)) = CM_{A_i}(x). \end{aligned}$$

Similarly,

$$\begin{aligned} CN_{g[f(A_i)]}(z) &= CN_{f(A_i)}(g^{-1}(z)) = CN_{f(A_i)}(y) \\ &= CN_{A_i}(f^{-1}(y)) = CN_{A_i}(x). \end{aligned}$$

Also,

$$\begin{aligned} CM_{[gf](A_i)}(z) &= CM_{g(f(A_i))}(z) = CM_{f(A_i)}(g^{-1}(z)) \\ &= CM_{f(A_i)}(y) = CM_{A_i}(f^{-1}(y)) \\ &= CM_{A_i}(x)\,\forall x \in X. \end{aligned}$$

Similarly,

$$\begin{aligned} CN_{[gf](A_i)}(z) &= CN_{g(f(A_i))}(z) = CN_{f(A_i)}(g^{-1}(z)) \\ &= CN_{f(A_i)}(y) = CN_{A_i}(f^{-1}(y)) \\ &= CN_{A_i}(x)\,\forall x \in X. \end{aligned}$$

Hence $g[f(A_i)] = [gf](A_i)$.

(iii) Similar to (ii).

□

Proposition 8. *Let X and Y be groups and $f\colon X \to Y$ be an isomorphism. Then, $A \in IFMG(X)$ if and only if $f(A) \in IFMG(Y)$.*

Proof. Suppose $A \in IFMG(X)$. Let $x, y \in Y$, then, $\exists f(a) = x$ and $f(b) = y$ since f is an isomorphism $\forall a, b \in X$. We know that

$$CM_B(x) = CM_A(f^{-1}(x)) = \bigvee_{a \in f^{-1}(x)} CM_A(a)$$

and

$$CN_B(x) = CN_A(f^{-1}(x)) = \bigwedge_{a \in f^{-1}(x)} CN_A(a).$$

Also,

$$CM_B(y) = CM_A(f^{-1}(y)) = \bigvee_{a \in f^{-1}(y)} CM_A(b),$$

and

$$CN_B(y) = CN_A(f^{-1}(y)) = \bigwedge_{a \in f^{-1}(y)} CN_A(b).$$

Clearly, $a \in f^{-1}(x) \neq \emptyset$ and $b \in f^{-1}(y) \neq \emptyset$. For $a \in f^{-1}(x)$ and $b \in f^{-1}(y) \implies x = f(a)$ and $y = f(b)$. Thus,

$$f(ab^{-1}) = f(a)f(b^{-1}) = f(a)(f(b))^{-1} = xy^{-1}.$$

Let $c = ab^{-1} \implies c \in f^{-1}(xy^{-1})$.

Now,

$$\begin{aligned} CM_B(xy^{-1}) &= \bigvee_{c \in f^{-1}(xy^{-1})} CM_A(c) = CM_A(ab^{-1}) \\ &\geq CM_A(a) \wedge CM_A(b) = CM_{f^{-1}(B)}(a) \wedge CM_{f^{-1}(B)}(b) \\ &= CM_B(f(a)) \wedge CM_B(f(b)) \\ &= CM_B(x) \wedge CM_B(y)\,\forall x, y \in Y. \end{aligned}$$

Similarly,

$$\begin{aligned} CN_B(xy^{-1}) &= \bigwedge_{c \in f^{-1}(xy^{-1})} CN_A(c) = CN_A(ab^{-1}) \\ &\le CN_A(a) \vee CN_A(b) = CN_{f^{-1}(B)}(a) \vee CN_{f^{-1}(B)}(b) \\ &= CN_B(f(a)) \vee CN_B(f(b)) \\ &= CN_B(x) \vee CN_B(y)\ \forall x, y \in Y. \end{aligned}$$

Hence, $f(A) \in IFMG(Y)$.
Conversely, let $a, b \in X$ and suppose $f(A) \in IFMG(Y)$. Then,

$$\begin{aligned} CM_A(ab^{-1}) &= CM_{f^{-1}(B)}(ab^{-1}) = CM_B(f(ab^{-1})) \\ &= CM_B(f(a)f(b^{-1})) = CM_B(f(a)(f(b))^{-1}) \\ &\ge CM_B(f(a)) \wedge CM_B(f(b)) \\ &= CM_{f^{-1}(B)}(a) \wedge CM_{f^{-1}(B)}(b) \\ &= CM_A(a) \wedge CM_A(b). \end{aligned}$$

Similarly,

$$\begin{aligned} CN_A(ab^{-1}) &= CN_{f^{-1}(B)}(ab^{-1}) = CN_B(f(ab^{-1})) \\ &= CN_B(f(a)f(b^{-1})) = CN_B(f(a)(f(b))^{-1}) \\ &\le CN_B(f(a)) \vee CN_B(f(b)) \\ &= CN_{f^{-1}(B)}(a) \vee CN_{f^{-1}(B)}(b) \\ &= CN_A(a) \vee CN_A(b). \end{aligned}$$

Hence, $A \in IFMG(X)$. □

Proposition 9. *Let X and Y be groups and $f\colon X \to Y$ be an isomorphism. Then, $B \in IFMG(X)$ if and only if $f^{-1}(B) \in IFMG(Y)$.*

Proof. Suppose $B \in IFMG(Y)$. Since $f^{-1}(B)$ is an inverse image of B, then we get

$$\begin{aligned} CM_{f^{-1}(B)}(ab^{-1}) &= CM_B(f(ab^{-1})) = CM_B(f(a)f(b^{-1})) \\ &= CM_B(f(a)(f(b))^{-1}) \ge CM_B(f(a)) \wedge CM_B(f(b)) \\ &= CM_{f^{-1}(B)}(a) \wedge CM_{f1(B)}(b), \end{aligned}$$

and

$$\begin{aligned} CN_{f^{-1}(B)}(ab^{-1}) &= CN_B(f(ab^{-1})) = CN_B(f(a)f(b^{-1})) \\ &= CN_B(f(a)(f(b))^{-1}) \le CM_B(f(a)) \vee CN_B(f(b)) \\ &= CN_{f^{-1}(B)}(a) \vee CN_{f1(B)}(b)\ a, b \in X. \end{aligned}$$

Hence $f^{-1}(B) \in IFMG(X)$.
Conversely, suppose $f^{-1}(B) \in IFMG(X)$. We get

$$\begin{aligned} CM_B(xy^{-1}) &= CM_{f(A)}(xy^{-1}) = CM_A(f^{-1}(xy^{-1})) \\ &= CM_A(f^{-1}(x)f^{-1}(y^{-1})) = CM_A(f^{-1}(x)(f^{-1}(y))^{-1}) \\ &\ge CM_A(f^{-1}(x)) \wedge CM_A(f^{-1}(y)) = CM_{f(A)}(x) \wedge CM_{f(A)}(y) \\ &= CM_B(x) \wedge CM_B(y), \end{aligned}$$

and

$$\begin{aligned} CN_B(xy^{-1}) &= CN_{f(A)}(xy^{-1}) = CN_A(f^{-1}(xy^{-1})) \\ &= CN_A(f^{-1}(x)f^{-1}(y^{-1})) = CN_A(f^{-1}(x)(f^{-1}(y))^{-1}) \\ &\leq CN_A(f^{-1}(x)) \vee CN_A(f^{-1}(y)) = CN_{f(A)}(x) \vee CN_{f(A)}(y) \\ &= CN_B(x) \vee CN_B(y)\ \forall x, y \in Y. \end{aligned}$$

Hence, $B \in IFMG(Y)$.
□

Corollary 2. *Let X and Y be groups and $f : X \to Y$ be an isomorphism. Then, the following statements hold;*

(i) $A^{-1} \in IFMG(X)$ *and if and only if* $f(A^{-1}) \in IFMG(Y)$.
(ii) $B^{-1} \in IFMG(Y)$ *and if and only if* $f^{-1}(B^{-1}) \in IFMG(X)$.

Proof. Straightforward from Propositions 8 and 9. □

Corollary 3. *Let X and Y be groups and $f : X \to Y$ be an isomorphism. If $\bigcap_{i \in I} A_i \in IFMG(X)$ and $\bigcap_{i \in I} B_i \in IFMG(Y)$. Then,*

(i) $f(\bigcap_{i \in I} A_i) \in IFMG(Y)$.
(ii) $f^{-1}(\bigcap_{i \in I} B_i) \in IFMG(X)$.

Proof. Straightforward from Propositions 8 and 9. □

Corollary 4. *Let X and Y be groups and $f : X \to Y$ be an isomorphism. If $\bigcup_{i \in I} A_i \in IFMG(X)$ and $\bigcup_{i \in I} B_i \in IFMG(Y)$. Then,*

(i) $f(\bigcup_{i \in I} A_i) \in IFMG(Y)$.
(ii) $f^{-1}(\bigcup_{i \in I} B_i) \in IFMG(X)$.

Proof. Straightforward from Propositions 3. □

Proposition 10. *Let f be a homomorphism of an abelian group X onto an abelian group Y. Let A and B be intuitionistic fuzzy multigroups of X such that $A \subseteq B$. Then, $f(N(A)) \subseteq N(f(A))$.*

Proof. Let $\in f(N(A))$. Then, $\exists u \in N(A)$ such that $f(u) = x$. For all $y, z \in Y$, we have

$$\begin{aligned} CM_{f(A)}(xyx^{-1}) &= CM_{A(f^{-1}}(xyx^{-1})) \\ &= CM_A(f^{-1}(x)f^{-1}(y)f^{-1}(x^{-1})) \\ &= CM_A(f^{-1}(x)f^{-1}(y)f^{-1}(x)^{-1}) \\ &= CM_A(f^{-1}(x)f^{-1}(y)(f^{-1}(x))^{-1}) \\ &= CM_A(f^{-1}(f(u))f^{-1}(f(v))(f^{-1}(f(u)))^{-1}) \\ &= CM_A(uvu^{-1}) = CM_A(vuu^{-1}) = CM_A(v) \\ &= CM_A(f^{-1}(y)) = CM_{f(A)}(y), \end{aligned}$$

and similarly,

$$\begin{aligned} CN_{f(A)}(xyx^{-1}) &= CN_{A(f^{-1}}(xyx^{-1})) \\ &= CN_A(f^{-1}(x)f^{-1}(y)f^{-1}(x^{-1})) \\ &= CN_A(f^{-1}(x)f^{-1}(y)f^{-1}(x)^{-1}) \\ &= CN_A(f^{-1}(x)f^{-1}(y)(f^{-1}(x))^{-1}) \\ &= CN_A(f^{-1}(f(u))f^{-1}(f(v))(f^{-1}(f(u)))^{-1}) \\ &= CN_A(uvu^{-1}) = CN_A(vuu^{-1}) = CN_A(v) \\ &= CN_A(f^{-1}(y)) = CN_{f(A)}(y), \end{aligned}$$

where $v \in X$ such that $f(v) = y$. Thus, $x \in N(f(A))$. Hence $f(N(A)) \subseteq N(f(A))$. $\square$

Proposition 11. *Let $f\colon X \to Y$ be a homomorphism of abelian groups X and Y . Let A and B be intuitionistic fuzzy multigroups of Y such that $B \subseteq A$. Then, $f^{-1}(N(B)) = N(f^{-1}(B))$.*

Proof. Let $x \in f^{-1}(N(B))$. Then for all $y \in X$,

$$\begin{aligned} CM_{f^{-1}(B)}(xyx^{-1}) &= CM_B(f(xyx^{-1})) = CM_B(f(x)f(y)f(x^{-1})) \\ &= CM_B(f(x)f(y)(f(x))^{-1}) = CM_B(f(y)f(x)(f(x))^{-1}) \\ &= CM_B(f(y)) = CM_{f^{-1}(B)}(y). \end{aligned}$$

Similarly,

$$\begin{aligned} CN_{f^{-1}(B)}(xyx^{-1}) &= CN_B(f(xyx^{-1})) = CN_B(f(x)f(y)f(x^{-1})) \\ &= CN_B(f(x)f(y)(f(x))^{-1}) = CN_B(f(y)f(x)(f(x))^{-1}) \\ &= CN_B(f(y)) = CN_{f^{-1}(B)}(y). \end{aligned}$$

Thus $x \in N(f^{-1}(B))$. So $f^{-1}(N(B)) \subseteq N(f^{-1}(B))$. Again, let $x \in N(f^{-1}(B))$ and $f(x) = u$. Then for all $v \in Y$,

$$\begin{aligned} CM_B(uvu^{-1}) &= CM_B(f(x)f(y)(f(x))^{-1}) = CM_B(f(y)f(x)(f(x))^{-1}) \\ &= CM_B(f(y)) = C_B(v), \end{aligned}$$

and

$$\begin{aligned} CN_B(uvu^{-1}) &= CN_B(f(x)f(y)(f(x))^{-1}) = CN_B(f(y)f(x)(f(x))^{-1}) \\ &= CN_B(f(y)) = CN_B(v), \end{aligned}$$

where $y \in X$ such that $f(y) = v$. Clearly, $u \in N(B)$, that is, $x \in f^{-1}(N(B))$. Thus $N(f^{-1}(B)) \subseteq f^{-1}(N(B))$. Hence $f^{-1}(N(B)) = N(f^{-1}(B))$. $\square$

Proposition 12. *Let $f : X \to Y$ be an isomorphism and let A be a normal sub-intuitionistic fuzzy multigroup of $B \in IFMG(X)$. Then, $f(A)$ is a normal sub-intuitionistic fuzzy multigroup of $f(B) \in IFMG(Y)$.*

Proof. By Proposition 8, $f(A), f(B) \in IFMG(Y)$ and so, $f(A) \subseteq f(B)$. We show that $f(A)$ is a normal sub-intuitionistic fuzzy multigroup of $f(B)$. Let $x, y \in Y$. Since f is an isomorphism, then for some $a \in X$ we have $f(a) = x$. Thus,

$$\begin{aligned} CM_{f(A)}(xyx^{-1}) &= CM_A(b) \text{ for } f(b) = xyx^{-1}, \forall b \in X \\ &= CM_A(a^{-1}ba) \text{ for } f(a^{-1}ba) = y \\ &\geq CM_A(b) \text{ for } f(b) = y, \forall a^{-1}ba \in X \\ &= CM_A(f^{-1}(y)) \text{ for } f(b) = y \\ &= CM_{f(A)}(y). \end{aligned}$$

Similarly,

$$\begin{aligned} CN_{f(A)}(xyx^{-1}) &= CN_A(b) \text{ for } f(b) = xyx^{-1}, \forall b \in X \\ &= CN_A(a^{-1}ba) \text{ for } f(a^{-1}ba) = y \\ &\leq CN_A(b) \text{ for } f(b) = y, \forall a^{-1}ba \in X \\ &= CN_A(f^{-1}(y)) \text{ for } f(b) = y \\ &= CN_{f(A)}(y). \end{aligned}$$

Hence, $f(A)$ is a normal sub-intuitionistic fuzzy multigroup of $f(B)$. $\square$

Proposition 13. *Let Y be a group and $A \in IFMG(Y)$. If f is an isomorphism of X onto Y and B is a normal sub-intuitionistic fuzzy multigroup of A, then $f^{-1}(B)$ is a normal sub-intuitionistic fuzzy multigroup of $f^{-1}(A)$.*

Proof. By Proposition 9, $f^{-1}(A), f^{-1}(B) \in IFMG(X)$. Since B is an intuitionistic fuzzy submultigroup of A, so $f^{-1}(B) \subseteq f^{-1}(A)$. Let $a, b \in X$, then we have

$$\begin{aligned} CM_{f^{-1}(B)}(aba^{-1}) &= CM_B(f(aba^{-1})) = CM_B(f(a)f(b)(f(a))^{-1}) \\ &= CM_B(f(a)(f(a))^{-1}f(b)) \geq CM_B(e) \wedge CM_B(f(b)) \\ &= CM_{f^{-1}(B)}(b), \end{aligned}$$

$\Longrightarrow CM_{f^{-1}(B)}(aba^{-1}) \geq CM_{f^{-1}(B)}(b)$.
Similarly,

$$\begin{aligned} CN_{f^{-1}(B)}(aba^{-1}) &= CN_B(f(aba^{-1})) = CN_B(f(a)f(b)(f(a))^{-1}) \\ &= CN_B(f(a)(f(a))^{-1}f(b)) \leq CM_B(e) \vee CN_B(f(b)) \\ &= CN_{f^{-1}(B)}(b), \end{aligned}$$

$\Longrightarrow CN_{f^{-1}(B)}(aba^{-1}) \leq CN_{f^{-1}(B)}(b)$. This completes the proof. □

4. Conclusion

In this paper, we have introduced the concept of homomorphism in intuitionistic fuzzy multigroups context and investigated some homomorphic properties of intuitionistic fuzzy multigroups. It was established that the homomorphic image and homomorphic preimage of intuitionistic fuzzy multigroups are also intuitionistic fuzzy multigroups. More theoretic concepts in group theory could be instituted in intuitionistic fuzzy multigroup setting in future research.

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