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Wronskian Representation of the Solutions to the KdV Equation

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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ABSTRACT

A method to construct solutions to the Korteweg-de-Vries (KdV) equation in terms of wronskians is given.
For this, a particular type of polynomials is considered and we obtain for each positive integer n, rational solutions in terms of determinants of order n.
Explicit solutions can be easily constructed and rational solutions from order 1 until order 10 are given.

Keywords: Derivatives; polynomials; positive integer; rational solutions.

1 INTRODUCTION

In the following, the Korteweg-de-Vries (KdV) equation

$$u_t + \frac{3}{2}uu_x + u_{xxx} = 0, \tag{1}$$

is considered, with the usual notations, where the subscripts x and t denote partial derivatives.

Korteweg and de Vries [1] introduced this equation (1) for the first time in 1895. This KdV equation is the basis of the most common tool for the 2-dimensional modelling of shallow water waves.

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Numerical modelling done by Zabusky and Kruskal revealed the presence of soliton solutions of this equation [2]. The inverse scattering technique developed in a work published in 1967 br Gardner et al. for the KdV equation enabled the derivation of analytic solutions for given initial conditions with zeros at infinity [3]. Being the historic first among the integrable nonlinear evolution equations, the KdV equation attracted significant attention from both physicists and mathematicians. Zakharov and Faddeev in 1971 [4] proved that this equation is a completely integrable [5].

Solutions were built by Hirota in 1971 by using the bilinear method [6].

Solutions in terms of Riemann theta functions [7] were construced by Its and Matveev in 1975. In the same year the expressions of periodic and almost periodic solutions [8] were given by Lax. Among different works realized, we can mention for example Airault et al. in 1977 [9], Adler and Moser in 1978 [10], Ablowitz and Cornille in 1979 [11], Freeman and Nimmo in 1984 [12], Matveev in 1992 [13], Ma in 2004 [14], Kovalyov in 2005 [15] and more recently Ma in 2015 [16].

Considering certain polynomials, we construct rational solutions in terms of wronkians of order n. So we get a very efficient method to construct rational solutions to the KdV equation. We give explicit solutions in the simplest cases for orders $n = 1$ until 10.

2 SOLUTIONS TO THE KdV EQUATION

Let p_n be the polynomials defined by

$$\left\{ \begin{array}{l} p_{3k}(x,t) = \sum_{l=0}^{k} \frac{x^{3l}}{(3l)!} \frac{(-t)^{k-l}}{(k-l)!}, \quad k \geq 0, \\ p_{3k+1}(x,t) = \sum_{l=0}^{k} \frac{x^{3l+1}}{(3l+1)!} \frac{(-t)^{k-l}}{(k-l)!}, \quad k \geq 0, \\ p_{3k+2}(x,t) = \sum_{l=0}^{k} \frac{x^{3l+2}}{(3l+2)!} \frac{(-t)^{k-l}}{(k-l)!}, \quad k \geq 0, \\ p_n(x,t) = 0, \quad n < 0. \end{array} \right. \tag{2}$$

Let $W_n(x,t) = W(p_{2n}, \ldots, p_{n+1})$ be the classical wronskian

$$W_n(x,t) = \det(\partial_x^{i-1}(p_{2n+1-j})_{\{1 \leq i \leq n,\, 1 \leq j \leq n\}}) \tag{3}$$

with $\partial_x^0(p_{n+j})$ meaning p_{n+j}.

We get the following statement

Theorem 2.1. *The function $v_n(x,t)$ expressed as*

$$v_n(x,t) = 4\partial_x^2(\ln(W_n(x,t))) \tag{4}$$

is a rational solution to the (KdV) equation (1)

$$u_t + \frac{3}{2}uu_x + u_{xxx} = 0. \tag{5}$$

Proof : The function $v_n(x,t) = 4\partial_x^2(\ln f(x,t))$ is a solution to the KdV equation if f satisfy the following equation

$$f_{xt}f^3 - f_x f_t f^2 + f_{4x}f^3 - 4f_{3x}f_x f^2 - 3f_x^4 + 6f_{2x}f_x^2 f = 0. \tag{6}$$

We have to verify (6) for $f = \det(p_{2n+2-i-j}(x,t))_{\{1 \leq i \leq n,\, 1 \leq j \leq n\}}$.
The proof is similar to this given in a previous paper [17] and we refer the interested reader to this paper.

3 RATIONAL SOLUTIONS TO THE KdV EQUATION

We construct explicitly rational solutions to the KdV equation.

We call rational solution to the KdV equation of order k, the following function

$$v_k(x,t) = 4\partial_x^2(\ln(W_k(x,t))).$$

In the following, we give some examples of these solutions.

3.1 Solutions of order 1

Example 3.1. *The function* $v_k(x,t)$ *expressed as*

$$v_k(x,t) = -\frac{8}{x^2} \tag{7}$$

is a rational solution to the KdV equation.

3.2 Solutions of order 2

Example 3.2. *The function* $v_k(x,t)$ *expressed as*

$$v_k(x,t) = 24\,\frac{\left(-x^3 + 24\,t\right)x}{x^6 + 24\,tx^3 + 144\,t^2} \tag{8}$$

is a rational solution to the KdV equation.

3.3 Solutions of order 3

Example 3.3. *The function* $v_k(x,t)$ *expressed as*

$$v_k(x,t) = -48\,\frac{\left(x^9 + 5400\,t^2x^3 + 43200\,t^3\right)x}{x^{12} + 120\,tx^9 + 2160\,t^2x^6 - 86400\,t^3x^3 + 518400\,t^4} \tag{9}$$

is a rational solution to the KdV equation.

3.4 Solutions of order 4

Example 3.4. *The function* $v_k(x,t)$ *expressed as*

$$v_k(x,t) = \frac{n(x,t)}{d(x,t)} \tag{10}$$

with $n(x,t) = -80(9144576000\,t^6 - 152409600\,t^4x^6 - 2116800\,t^3x^9 + 22680\,t^2x^{12} + 144\,tx^{15} + x^{18})$
and
$d(x,t) = (x^{18} + 360\,tx^{15} + 32400\,t^2x^{12} + 604800\,t^3x^9 + 108864000\,t^4x^6 + 91445760000\,t^6)x^2$
is a rational solution to the KdV equation.

3.5 Solutions of order 5

Example 3.5. *The function $v_k(x,t)$ expressed as*

$$v_k(x,t) = \frac{n(x,t)}{d(x,t)} \tag{11}$$

with

$n(x,t) = 120\,(-x^{27} - 504\,tx^{24} - 120960\,t^2x^{21} - 1447891200\,t^4x^{15}$
$- 512096256000\,t^5x^{12} - 17667320832000\,t^6x^9 - 184354652160000\,t^7x^6$
$- 19357238476800000\,t^8x^3 + 154857907814400000\,t^9)x$

and

$d(x,t) = x^{30} + 840\,tx^{27} + 226800\,t^2x^{24} + 25401600\,t^3x^{21}$
$+ 1905120000\,t^4x^{18} - 109734912000\,t^5x^{15} - 9601804800000\,t^6x^{12}$
$- 1152216576000000\,t^7x^9 + 58071715430400000\,t^8x^6$
$+ 774289539072000000\,t^9x^3 + 2322868617216000000\,t^{10}$

is a rational solution to the KdV equation.

3.6 Solutions of order 6

Example 3.6. *The function $v_k(x,t)$ expressed as*

$$v_k(x,t) = \frac{n(x,t)}{d(x,t)} \tag{12}$$

with

$n(x,t) = -168\,(x^{39} + 1200\,tx^{36} + 604800\,t^2x^{33} + 120960000\,t^3x^{30} + 23278752000\,t^4x^{27} + 2112397056000\,t^5x^{24} -$
$1594448271360000\,t^6x^{21} - 294045670195200000\,t^7x^{18}$
$- 172929272592384000000\,t^8x^{15} - 7495122738216960000000\,t^9x^{12}$
$+ 668939704385863680000000\,t^{10}x^9 - 40473662786371584000000000\,t^{11}x^6$
$+ 708289098761502720000000000\,t^{12}x^3 + 33997876740552130560000000000\,t^{13})x$

and

$d(x,t) = x^{42} + 1680\,tx^{39} + 1038240\,t^2x^{36} + 313286400\,t^3x^{33} + 53317958400\,t^4x^{30}$
$+ 3990083328000\,t^5x^{27} + 441683020800000\,t^6x^{24} + 93283453992960000\,t^7x^{21}$
$+ 18418412410675200000\,t^8x^{18} - 534879213590937600000\,t^9x^{15}$
$+ 81739423771213824000000\,t^{10}x^{12} + 1015214374891487232000000\,t^{11}x^9$
$+ 138824663357254533120000000\,t^{12}x^6 - 2379851371838649139200000000\,t^{13}x^3$
$+ 71395541155159474176000000000\,t^{14}$

is a rational solution to the KdV equation.

3.7 Solutions of order 7

Example 3.7. *The function $v_k(x,t)$ defined by*

$$v_k(x,t) = \frac{n(x,t)}{d(x,t)} \tag{13}$$

with

$n(x,t) = -224\,x^{54} - 532224\,tx^{51} - 533433600\,t^2x^{48} - 274743705600\,t^3x^{45} - 90973899878400\,t^4x^{42} - 17237159976960000\,t^5x^{39} -$
$3411605741322240\,00\,t^6x^{36} - 1094458263478272000000\,t^7x^{33}$
$- 74147750224632741888\,0000\,t^8x^{30} - 1540208748065346551808\,00000\,t^9x^{27}$
$- 1603439027558265087590\,4000000\,t^{10}x^{24} - 1779380872847017409249\,28000000\,t^{11}x^{21}$
$- 9399603769296238959132\,6720000000\,t^{12}x^{18} - 4714367526984320747757\,895680000000\,t^{13}x^{15}$
$- 8629203804459814125456\,85094400000000\,t^{14}x^{12}$
$+ 2060306024630715310574\,1938688000000000\,t^{15}x^{9}$
$+ 1483420337734115023613\,41958553600000000\,t^{16}x^{6}$
$- 2966840675468230047226\,8391710720000000000\,t^{18}$

and

$d(x,t) = (x^{54} + 3024\,tx^{51} + 3646944\,t^2x^{48} + 2336947200\,t^3x^{45} + 895482604800\,t^4x^{42}$
$+ 211662185011200\,t^5x^{39} + 34736257188864000\,t^6x^{36} + 3672731815796736000\,t^7x^{33}$
$- 240312372794081280000\,t^8x^{30} - 285039517734390988800000\,t^9x^{27}$
$- 266068799949758496768\,00000\,t^{10}x^{24} - 500533740312778742169\,6000000\,t^{11}x^{21} -$
$341169543038359143972\,864000000\,t^{12}x^{18} + 130784494154255779579\,8220800000000\,t^{13}x^{15}$
$+ 114398146526029352864\,31744000000000\,t^{14}x^{12}$
$+ 735823580225255468062\,212096000000000\,t^{15}x^{9}$
$+ 132448244440545984251\,19817728000000000\,t^{16}x^{6}$
$+ 370855084433528755903\,354896384000000000000\,t^{18})x^2$

is a rational solution to the KdV equation.

3.8 Solutions of order 8

Example 3.8. *The function $v_k(x,t)$ expressed as*

$$v_k(x,t) = \frac{n(x,t)}{d(x,t)} \tag{14}$$

with

$n(x,t) = 288\,(-x^{69} - 4200\,tx^{66} - 7623000\,t^2x^{63} - 7730553600\,t^3x^{60} - 5000876496000\,t^4x^{57}$
$- 2135257886361600\,t^5x^{54} - 609838468081920000\,t^6x^{51} - 163518036732610560000\,t^7x^{48}$
$- 25169090600103690240000\,t^8x^{45} + 2413029469529708789760000\,0\,t^9x^{42}$
$+ 172682250259749832227225600\,00\,t^{10}x^{39} + 503505298758686413219430400\,0000\,t^{11}x^{36}$
$+ 705713595558650622946050048\,000000\,t^{12}x^{33} + 831840770951738870078531174\,40000000\,t^{13}x^{30}$

$$+\,34661159542275399464361866035200000000\,t^{14}x^{27}$$
$$-\,515161125869404732022375621591040000000\,t^{15}x^{24}$$
$$-\,28979499285932207292849826257960960000000\,t^{16}x^{21}$$
$$-\,29698075161436982772740660102430720000000000\,t^{17}x^{18}$$
$$+\,219447562532189704845715806256889856000000000\,t^{18}x^{15}$$
$$-\,216142591024938360620006524225490780160000000000\,t^{19}x^{12}$$
$$-\,546566345078602374647796382591145213952000000000000\,t^{20}x^{9}$$
$$-\,366925658234586209553765403697552031744000000000000\,t^{21}x^{6}$$
$$-\,2407949632164472000196585461765185208320000000000000000\,t^{22}x^{3}$$
$$+\,11558158234389465600943610216472888999936000000000000000\,t^{23})x$$

and

$$d(x,t) = x^{72} + 5040\,tx^{69} + 10674720\,t^{2}x^{66} + 12666931200\,t^{3}x^{63} + 9461359353600\,t^{4}x^{60}$$
$$+\,4710504608409600\,t^{5}x^{57} + 1632339334112256000\,t^{6}x^{54} + 397315117938106368000\,t^{7}x^{51}$$
$$+\,65431718836877352960000\,t^{8}x^{48} + 9437611212912434872320000\,t^{9}x^{45}$$
$$+\,468502097720025168543744000\,t^{10}x^{42} + 163198234161769093524357120000\,t^{11}x^{39}$$
$$+\,41945348556753657320869724160000\,t^{12}x^{36} + 5629952914038054932620876185600000\,t^{13}x^{33}$$
$$-\,4653435919389780491390674796544000000\,t^{14}x^{30}$$
$$+\,123438329975214472796599077568512000000\,t^{15}x^{27}$$
$$+\,7237700541575855582398849996554240000000000\,t^{16}x^{24}$$
$$+\,630659838963943927023985589175189504000000000\,t^{17}x^{21}$$
$$+\,46320512087509853987540360999762657280000000000\,t^{18}x^{18}$$
$$-\,669581287754090052732145991429432279040000000000\,t^{19}x^{15}$$
$$-\,52328602395955192385224497913684977254400000000000000\,t^{20}x^{12}$$
$$-\,614600477542931901002557051193399653171200000000000000\,t^{21}x^{9}$$
$$+\,15025605704706305281226693281414755699916800000000000000\,t^{22}x^{6}$$
$$+\,138697898812673587211323322597674667999232000000000000000\,t^{23}x^{3}$$
$$+\,2773957976253471744226466451953493359984640000000000000000\,t^{24}$$

is a rational solution to the KdV equation.

3.9 Solutions of order 9

Example 3.9. *The function $v_k(x,t)$ defined by*

$$v_k(x,t) = \frac{n(x,t)}{d(x,t)} \tag{15}$$

with

$$n(x,t) = -360\,(x^{87} + 6864\,tx^{84} + 20845440\,t^{2}x^{81} + 36883123200\,t^{3}x^{78} + 42793643692800\,t^{4}x^{75}$$
$$+\,34479450061056000\,t^{5}x^{72} + 20002051134793728000\,t^{6}x^{69} + 8731536922920075264000\,t^{7}x^{66}$$
$$+\,2751355428529975787520000\,t^{8}x^{63} + 497883367475949628293120000\,t^{9}x^{60}$$

$$+ 2178302642158043732941209600000\, t^{10} x^{57} + 32570625977159758431947587584 0000\, t^{11} x^{54}$$

$$+ 236136684508248983555307126784 00000\, t^{12} x^{51} + 9084218005546696550259477813657 6000000\, t^{13} x^{48}$$

$$+ 2246458626379902791685372161871052 8000000\, t^{14} x^{45} + 34711479670808115332009109650606653 44000000\, t^{15} x^{42}$$

$$+ 305398496769968754512043060254019158016000000\, t^{16} x^{39}$$

$$+ 17506664385513663140183161643807413370880000000\, t^{17} x^{36}$$

$$+ 231129923632261115604054675708605129293824000000000\, t^{18} x^{33}$$

$$+ 389754549604414714891502174564268060652339200000 0000\, t^{19} x^{30}$$

$$+ 355735131990323431740828949883178812563469107200000000\, t^{20} x^{27}$$

$$+ 9378985656695796563336893013692224464703329402880000000 0\, t^{21} x^{24}$$

$$- 62860203600534130479725052582686904710851335290880000000000\, t^{22} x^{21}$$

$$- 162329335579032153111484405548792940444728667668480000000000000\, t^{23} x^{18}$$

$$- 759866371190120339768301069906197191390257007401369600000000000\, t^{24} x^{15}$$

$$- 2856556091043632926211314732491165804870086303730892800000000000 0\, t^{25} x^{12}$$

$$+ 13087464265664955125039994390898846051632948803209789440000000000 0\, t^{26} x^{9}$$

$$- 63454372197163418788072700077085314189735509348895948800000000000 00\, t^{27} x^{6}$$

$$+ 58298704456143891011541793195822132411819499214298152960000000000000 0\, t^{28} x^{3}$$

$$+ 19988127242106476918242900524281873969766685444902223872000000000000 00\, t^{29})x$$

and

$$d(x,t) = x^{90} + 7920\, t x^{87} + 27324000\, t^2 x^{84} + 54752544000\, t^3 x^{81} + 71622115488000\, t^4 x^{78} + 65066476598323200\, t^5 x^{75}$$

$$+ 42677705283333120000\, t^6 x^{72} + 20691357050667663360000\, t^7 x^{69} + 7527123164747777310720000\, t^8 x^{66}$$

$$+ 2107511386632847215820800000\, t^9 x^{63} + 469311169848026334959370240000\, t^{10} x^{60}$$

$$+ 40479537955403752535044915200000\, t^{11} x^{57} - 3864198318708114970289189683200000 0\, t^{12} x^{54}$$

$$- 2822971773306238911652212139622400000 0\, t^{13} x^{51} - 975524760175268043556099819831296000000 0\, t^{14} x^{48}$$

$$- 160541598368625163091998264272302899200000 0\, t^{15} x^{45}$$

$$- 3714542651165932086049816927293092659200000 00\, t^{16} x^{42}$$

$$- 8412855882694287761370587823486523146240000 0000\, t^{17} x^{39}$$

$$+ 2835203291469226997662845431796479840747520000 00000\, t^{18} x^{36}$$

$$+ 1312804996097541074607313701710634239498649600000 00000\, t^{19} x^{33}$$

$$+ 2430885024143419353984724033937664218012162457600000 00000\, t^{20} x^{30}$$

$$+ 14641066550792050053314339432420050447748195942400000 000000\, t^{21} x^{27}$$

$$+ 66022945062529841216233919963113985942670331084800000 00000000\, t^{22} x^{24}$$

$$+ 63406934390929600954364108816021535325829103353856000000 0000000\, t^{23} x^{21}$$

$$+ 1950865215423383423335825547841625228435453001072640000000 00000000\, t^{24} x^{18}$$

$$- 29476806265415574162291514021822848249977034812424192000000 000000000\, t^{25} x^{15}$$

$$+ 20280161568695694925165280564409368881662627277701120000000 0000000000\, t^{26} x^{12}$$

$$+ 257783387050976388826545344063159088895800506729889792000000 000000000000\, t^{27} x^{9}$$

$$+ 27483674957896405762583988220887576708429192486740557824000000 00000000000\, t^{28} x^{6}$$

$$- 2998219086315971537736435078642281095465002816735333580800000 000000000000000\, t^{29} x^{3}$$

$$+ 59964381726319430754728701572845621909300056334706671616000000 000000000000\, t^{30}$$

is a rational solution to the KdV equation.

3.10 Solutions of order 10

Example 3.10. *The function $v_k(x,t)$ expressed as*

$$v_k(x,t) = \frac{n(x,t)}{d(x,t)} \tag{16}$$

with

$$\begin{aligned}
n(x,t) = &-440\,x^{108} - 4656960\,tx^{105} - 22239360000\,t^2x^{102} - 63527473152000\,t^3x^{99} - 121936711792896000\,t^4x^{96}\\
&- 167286869336530944000\,t^5x^{93} - 170457089665294688256000\,t^6x^{90} - 132572493457040495738880000\,t^7x^{87}\\
&- 79773272960436964977868800000\,t^8x^{84} - 37509353281280939031016243200000\,t^9x^{81}\\
&- 14763698113159334472421736448000000\,t^{10}x^{78} - 4978722500676502355093319843840000000\,t^{11}x^{75}\\
&+ 307544827106787687424323357769728000000\,t^{12}x^{72} + 306218141634709752462884327254917120000000 0\,t^{13}x^{69}\\
&+ 2999970822894313674432005462201582223360000000\,t^{14}x^{66}\\
&+ 1673514712689379266786652586236034049638400000000\,t^{15}x^{63}\\
&+ 614189063528038366650760463291115511794892800000000\,t^{16}x^{60}\\
&+ 155794693835149410545599807873777332894931353600000000\,t^{17}x^{57}\\
&+ 29027683107370049595272518533072971960313446400000000000\,t^{18}x^{54}\\
&+ 5218610071634201355190994147633369369868352094208000000000\,t^{19}x^{51}\\
&+ 787866454542969747639992422970796318251475271680000000000000\,t^{20}x^{48}\\
&- 37854553858637413068475237418277708609060768520863744000000000000\,t^{21}x^{45}\\
&- 131329475080569841532200337554391930500452935693689159680000000000\,t^{22}x^{42}\\
&- 3513566821938187068323458047568782732506359185178361856000000000000\,t^{23}x^{39}\\
&- 121327715332732581262858555471852629639080434785494520299520000000000000\,t^{24}x^{36}\\
&+ 2595423786624385310311762738282885516779623225677141663285248000000000000\,t^{25}x^{33}\\
&- 16133260898172723024130520161952145057148588369685859622641991680000000000000\,t^{26}x^{30}\\
&- 386196572754775953267255313322889374931945882200169499242277109760000000000000\,t^{27}x^{27}\\
&- 3949639858292160818760942549919713001064650682276256769927135887360000000000000\,t^{28}x^{24}\\
&- 22398063092453206574593483244714613369097180484635029213910275194880000000000000 0\,t^{29}x^{21}\\
&- 959926062518528461851509539811921759575390823512978836699037682368512000000000000 0\,t^{30}x^{18}\\
&- 2443740619574356759064843703785097208419752494714286388156930420690124800000000000 000\,t^{31}x^{15}\\
&- 4608541109217035295960114787434198123641592451380593376086737533492264960000000000 0000\,t^{32}x^{12}\\
&+ 5199379712962809051852437196079608139493078663096054065328626960863068160000000000 00000\,t^{33}x^9\\
&+ 3743553393333222517333754781177317860435016637429158927036611411821409075200000000 000000\,t^{34}x^6\\
&- 3743553393333222517333754781177317860435016637429158927036611411821409075200000000 0000000\,t^{36}
\end{aligned}$$

and

$$\begin{aligned}
d(x,t) = (&x^{108} + 11880\,tx^{105} + 63082800\,t^2x^{102} + 199635321600\,t^3x^{99} + 423349568928000\,t^4x^{96}\\
&+ 640877607943372800\,t^5x^{93} + 720356494667217408000\,t^6x^{90} + 617087988787839528960000\,t^7x^{87}\\
&+ 410285748104906240286720000\,t^8x^{84} + 214683708353169921004339200000\,t^9x^{81}
\end{aligned}$$

$+ 891884499533276300486443008000000\, t^{10} x^{78}$
$+ 29191715419424489706841178112000000\, t^{11} x^{75}$
$+ 775168403895240598967724461260800000\, t^{12} x^{72}$
$+ 25393642893977347165865575301775360000000\, t^{13} x^{69}$
$+ 168994897506676823598108992801341440000000\, t^{14} x^{66}$
$+ 1130347213654414430863954455878455787520000000\, t^{15} x^{63}$
$+ 506556111685474474049446101072644721868800000000\, t^{16} x^{60}$
$+ 17020559905124457355727754138125310319656960000000\, t^{17} x^{57}$
$+ 423838149239441232144856200803466137233784832000000000\, t^{18} x^{54} +$
$177359494426191035376420187330956704562020352000000000\, t^{19} x^{51}$
$- 2007835263082516552782344585712077243130021150720000000000\, t^{20} x^{48}$
$+ 15707269194135167813143155649809134346239031115776000000000000\, t^{21} x^{45}$
$+ 98060016736581296781691171288377731242774346367762432000000000\, t^{22} x^{42}$
$+ 2354802714065562148570473679063743107392403519798928998400000000000\, t^{23} x^{39}$
$+ 3314466802974799576312045027363982121858355358032566681600000000000000\, t^{24} x^{36}$
$+ 32966193401700514519086283668796827258457603526842221645004800000000000\, t^{25} x^{33}$
$+ 283516231803993134625581454811050981056415186227296565880422400000000000\, t^{26} x^{30}$
$- 6892888800577414119786666618460383733780467006029045095936819200000000000\, t^{27} x^{27}$
$- 513772787482820792658121007265958978690507868109926231271958118400000000000\, t^{28} x^{24}$
$- 631540059369508598822797386451140237938001415764898497226667458560000000000000\, t^{29} x^{21}$
$- 7063991438379220991398898659394025436931764764094061323485227620761600000000000\, t^{30} x^{18}$
$+ 659321466503329426708727125293739852853633880943600167351297333264384000000000000\, t^{31} x^{15}$
$+ 5725198244302650719380471063365452626127627381183172848266189152976896000000000000000\, t^{32} x^{12}$
$+ 18569213260581460899472989985998600498189566653914478804745096288796672000000000000000\, t^{33} x^{9}$
$+ 334245838690466296190513819747974808967412199770460618485411733198340096000000000000000\, t^{34} x^{6}$
$+ 4679441741666528146667193476471647325543770796786448658795764264777676134400000000000000000\, t^{36})x^{2}$

is a rational solution to the KdV equation.

4 CONCLUSION

New representation of the solutions to the KdV equation in terms of wronskians have been constructed. Precisely, these solutions can be written as the second derivative with respect to x of a logarithm of a wronskian involving particular polynomials. Surprisingly, although the reprsentation of the solutions is different from this given in [17], we recover the same solutions.

In particular, rational solutions appear as the quotient of two polynomials in x and t; the numerator is a polynomial of degree $n(n+1)-2$ in x and the denominator is a polynomial of degree $n(n+1)$ in x have been construced. We get a very efficient method to construct rational solutions to the KdV equation.

Other recent works as [18], [19] or [20] can be quoted.

The solutions presented in this paper are different from these presented in a previous work of the author [21].

DISCLAIMER (ARTIFICIAL INTELLIGENCE)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of manuscripts.

COMPETING INTERESTS

Author has declared that no competing interests exist.

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