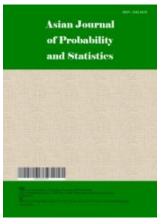

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A New Two-Parametric Maxwell-Rayleigh Distribution: Application to the Analysis of Engineering Data

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

In this study a new generalisation of Rayleigh Distribution has been studied and referred it is as "A New Two-Parametric Maxwell-Rayleigh Distribution". This distribution is obtained by adopting T-X family procedure. Several distributional properties of the formulated distribution including moments, moment generating function, Characteristics function and incomplete moments have been discussed. The expressions for ageing properties have been derived and discussed explicitly. The behaviour of the pdf and Hazard rate function has been illustrated through different graphs. The parameters are estimated through the technique of MLE. Eventually the versatility and the efficacy of the formulated distribution have been examined through real life data sets related to engineering science.

Keywords: *Maxwell distribution; Rayleigh distribution; Renyi entropy; maximum likelihood estimation; order statistics.*

1 Introduction

The Rayleigh distribution, named after the British physicist and mathematician Lord Rayleigh, is a one-parameter continuous and simplest velocity probability distribution. This distribution has a wide variety of

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applicability in different disciplines. The distribution's expansion and modification allow additional adaptability for analyzing real-world data. These extensions can be developed by either introducing new parameters or by adopting the compounding approach. This distribution finds its significance in numerous disciplines such as Physical Science, Engineering, Medical Sciences, and so on. For instance An extension of Rayleigh distribution and its properties by Kahkashan Ateeq, Tahira Bano Qasim and Ayesha Rehman Alvi [1], A generalized Rayleigh distribution and its applications by Lishamol Tomy and jiju Gillariose [2], Odd Lindley- Rayleigh distribution by Terna Godfrey Ieren [3], New generalization of Rayleigh distribution by A.A Bhat et al. [4], Top-Leone power Rayleigh distribution with properties and application related in engineering sciences by Aijaz et al. [5], Alpha-Power Exponentiated Inverse Rayleigh Distribution and its applications to real and simulated data(2021).

The probability density function (pdf) of Rayleigh distribution is stated as

$$g(x,\alpha)=\alpha x e^{-\frac{\alpha}{2}x^2} \qquad x>0,\alpha>0 \tag{1.1}$$

The related cumulative density function (cdf) is stated as

$$G(x,\alpha)=1-e^{\frac{-\alpha}{2}x^2} \qquad x>0,\alpha>0 \tag{1.2}$$

Maxwell-Boltzmann distribution, named after J.C.Maxwell and L.Boltzmann is a continuous probability distribution that forms the basis of the kinetic energy of gases, its fundamental properties like pressure and diffusion. This distribution is also known as the distribution of velocities, energy and magnitude of momenta of molecules.

The probability density function (pdf) of Maxwell distribution is stated as

$$h(t,\theta)=\sqrt{\frac{2}{\pi}}\frac{1}{\theta^3}t^2e^{-\frac{t^2}{2\theta^2}} \qquad t>0,\theta>0 \tag{1.3}$$

The related cumulative density function (cdf) is stated as

$$H(t,\theta)=\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{2\theta^2}t^2\right) \qquad t>0,\theta>0 \tag{1.4}$$

This distribution for the first time was considered as a lifetime model by Tyagi and Bhattacharya [6] and discussed the Bayes and minimum variance unbiased estimation procedures for its parameter and reliability function. After generalizing it by adding one more parameter, it’s classical and Bayes’ estimators were obtained by Chaturvedi and Rani [7]. Some more estimation for the Maxwell distribution was studied by Kazmi et al. [8], Dar et al. [9], Aijaz et al. [10] and so on.

2 Materials and Methods

Transformed-Transformer (T-X) family of distributions Alzaatreh et al. (2013)) is given by

$$\mathrm{F}(x,\zeta)=\int_0^{\frac{G(x,\varsigma)}{\overline{G}(x,\varsigma)}}h(t)dt \tag{1.5}$$

Using equation (1.3) in (1.5) we obtain the cdf as

$$F(x,\zeta)=\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{2\theta^{2}}\left(\frac{G(x,\zeta)}{\overline{G}(x,\zeta)}\right)^{2}\right)\qquad -\infty<x<\infty \tag{1.6}$$

The corresponding pdf of (1.6) becomes

$$f(x,\varsigma)=\frac{2g(x,\theta,\zeta)}{\theta^{2}\sqrt{2\pi}}\frac{[G(x,\varsigma)]^{2}}{[\overline{G}(x,\varsigma)]^{4}}e^{-\frac{1}{2\theta^{2}}\left[\frac{G(x,\varsigma)}{\overline{G}(x,\varsigma)}\right]^{2}}\qquad -\infty<x<\infty \tag{1.7}$$

Where $\overline{G}(x,\varsigma)=1-G(x,\varsigma)$

Survival Function, Hazard Rate Function are given by

$$S(x,\zeta)=1-\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{2\theta^{2}}\left(\frac{G(x,\zeta)}{\overline{G}(x,\zeta)}\right)^{2}\right)$$

$$h_{x}(x,\zeta)=\frac{\frac{2g(x,\theta,\zeta)}{\theta^{2}\sqrt{2\pi}}\frac{[G(x,\varsigma)]^{2}}{[\overline{G}(x,\varsigma)]^{4}}e^{-\frac{1}{2\theta^{2}}\left[\frac{G(x,\varsigma)}{\overline{G}(x,\varsigma)}\right]^{2}}}{1-\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{2\theta^{2}}\left(\frac{G(x,\zeta)}{\overline{G}(x,\zeta)}\right)^{2}\right)}$$

2.1 Mixture Form

From (1.7), we obtain the mixture form of the probability density function as

$$f(x,\theta,\varsigma)=\frac{2g(x,\theta,\zeta)}{\theta^{2}\sqrt{2\pi}}\frac{[G(x,\varsigma)]^{2}}{[\overline{G}(x,\varsigma)]^{4}}e^{-\frac{1}{2\theta^{2}}\left[\frac{G(x,\varsigma)}{\overline{G}(x,\varsigma)}\right]^{2}} \tag{1.8}$$

Using the following expression in (1.8)

$$e^{\frac{-1}{2\theta^{2}}\left(\frac{G(x,\varsigma)}{\overline{G}(x,\varsigma)}\right)^{2}}=\sum_{p=0}^{\infty}\frac{(-1)^{p}}{p!}\left(\frac{1}{2\theta^{2}}\right)^{p}\left(\frac{G(x,\theta)}{\overline{G}(x,\theta)}\right)^{2p}$$

After solving above expression we obtain

$$\begin{aligned}
f(x,\theta,\varsigma)&=\sqrt{\frac{2}{\pi}}\frac{g(x,\theta,\zeta)}{\theta^{2}}\frac{\left(G(x,\varsigma)\right)^{2}}{\left(\overline{G}(x,\varsigma)\right)^{4}}\sum_{p=0}^{\infty}\frac{(-1)^{p}}{p!}\left(\frac{1}{2\theta^{2}}\right)^{p}\left(\frac{G(x,\theta)}{\overline{G}(x,\theta)}\right)^{2p}\\
&=\sum_{p=0}^{\infty}\frac{(-1)^{p}}{p!}\sqrt{\frac{2}{\pi}}\frac{2g(x,\theta,\zeta)}{\left(2\theta^{2}\right)^{p+1}}\frac{\left(G(x,\theta)\right)^{2p+2}}{\left(\overline{G}(x,\theta)\right)^{2p+4}}\\
&=\sum_{p=0}^{\infty}\frac{(-1)^{p}}{p!}\sqrt{\frac{2}{\pi}}\frac{2g(x,\theta,\zeta)}{\left(2\theta^{2}\right)^{p+1}}\left(G(x,\theta)\right)^{2p+2}\left(1-G(x,\theta)\right)^{-(2p+4)}
\end{aligned}$$

We know the generalized binomial theorem

$$(1-Z)^{-a}=\sum_{q=0}^{\infty}\binom{a+q-1}{q}(Z)^{q}$$

$$f(x,\theta,\varsigma)=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(-1)^{p}}{p!}\binom{2p+q+3}{q}\sqrt{\frac{2}{\pi}}\frac{2}{\left(2\theta^{2}\right)^{p+1}}g(x,\zeta)\left(G(x,\zeta)\right)^{2p+q+2}$$

$$f(x,\theta,\varsigma)=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\delta_{p,q}\,g(x,\zeta)\left(G(x,\zeta)\right)^{2p+q+4} \tag{1.9}$$

Where $\delta_{p,q}=\dfrac{(-1)^{p}}{p!}\dbinom{2p+q+3}{q}\sqrt{\dfrac{2}{\pi}}\dfrac{2}{\left(2\theta^{2}\right)^{p+1}}$

2.2 The Maxwell-Rayleigh distribution

In this part we formulate the pdf and cdf of the Maxwell-Rayleigh distribution (MRD). Using equation (1.2) in equation (1.6), we obtain the cdf of the newly formulated distribution as

$$F(x,\alpha,\theta)=\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{2\theta^{2}}\left(e^{\frac{\alpha}{2}x^{2}}-1\right)^{2}\right)\qquad x>0,\theta,\alpha>0 \tag{1.10}$$

The related pdf of (1.10) is stated as

$$f(x,\alpha,\theta)=\sqrt{\frac{2}{\pi}}\frac{\alpha x}{\theta^{2}}e^{\frac{\alpha}{2}x^{2}}\left(e^{\frac{\alpha}{2}x^{2}}-1\right)^{2}e^{-\frac{1}{2\theta^{2}}\left[e^{\frac{\alpha}{2}x^{2}}-1\right]^{2}}\qquad x>0,\theta,\alpha>0 \tag{1.11}$$

Figs. (1.1),(1.2),(1.3) and (1.4) expounds some of possible contours of cdf and pdf for distinct choice of parameters respectively.

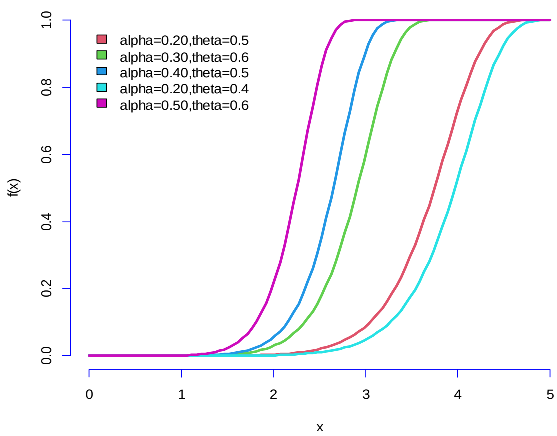

Fig 1.1 cdf of MRD under different values of parameters

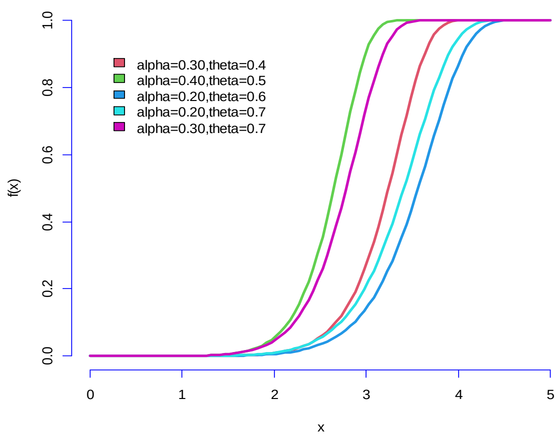

Fig 1.2 cdf of MRD under different values of parameters

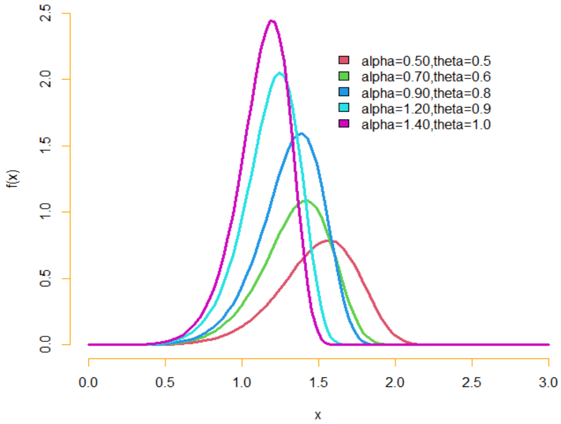

Fig 1.3 pdf of MRD under different values of parameters

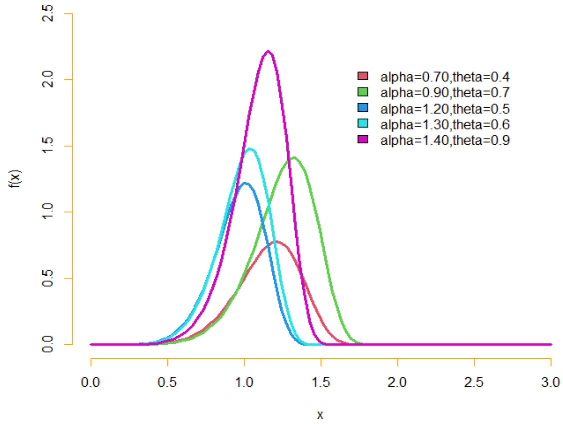

Fig 1.4 pdf of MRD under different values of parameters

3 Reliability Measures

The Survival function $S(x)$, Hazard rate function $h_x(x)$ and reverse hazard rate function $H_x(x)$ of Maxwell-Rayleigh distribution are given as

$$S(x,\alpha,\theta)=1-\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{2\theta^2}\left(e^{\frac{\alpha}{2}x^2}-1\right)^2\right) \tag{2.1}$$

$$h_x(x)=\frac{\sqrt{\frac{2}{\pi}\frac{\alpha x}{\theta^2}}e^{\frac{\alpha}{2}x^2}\left(e^{\frac{\alpha}{2}x^2}-1\right)^2 e^{-\frac{1}{2\theta^2}\left[e^{\frac{\alpha}{2}x^2}-1\right]^2}}{1-\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{2\theta^2}\left(e^{\frac{\alpha}{2}x^2}-1\right)^2\right)} \tag{2.2}$$

$$H_x(x)=\frac{\sqrt{\frac{2}{\pi}\frac{\alpha x}{\theta^2}}e^{\frac{\alpha}{2}x^2}\left(e^{\frac{\alpha}{2}x^2}-1\right)^2 e^{-\frac{1}{2\theta^2}\left[e^{\frac{\alpha}{2}x^2}-1\right]^2}}{\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{2\theta^2}\left(e^{\frac{\alpha}{2}x^2}-1\right)^2\right)} \tag{2.3}$$

Fig. (2.1) and (2.2) expounds some of possible contours of Hazard rate function for distinct choice of parameters.

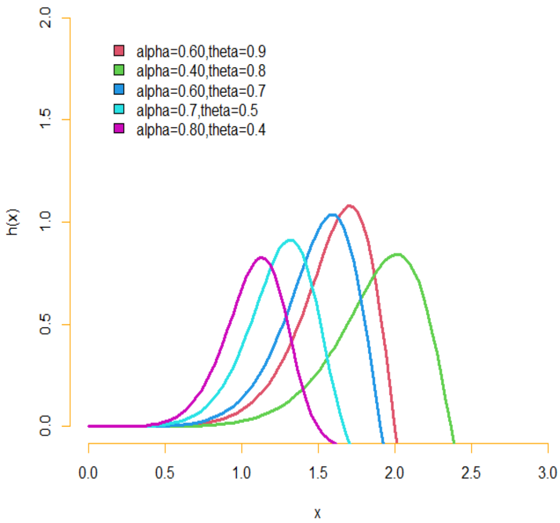

Fig 2.1 HRF of MRD under different values of parameters

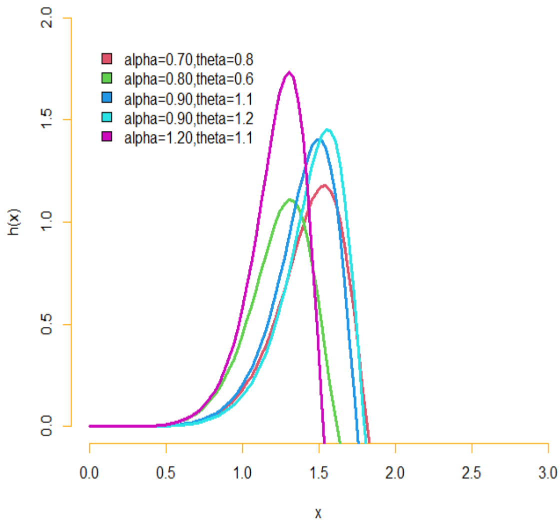

Fig 2.2 HRF of MRD under different values of parameters

4 Moments

We know that r^{th} moment which is denoted as μ_r' and is defined by

$$\mu_r' = \int_0^\infty x^r f(x,\varsigma)dx$$

$$= \int_0^\infty x^r \sum_{p=0}^\infty \sum_{q=0}^\infty \frac{(-1)^p}{p!}\binom{2p+q+3}{q}\sqrt{\frac{2}{\pi}}\frac{2}{(2\theta^2)^{p+1}} g(x,\zeta)(G(x,\zeta))^{2p+q+2}dx$$

$$= \sum_{p=0}^\infty \sum_{q=0}^\infty \delta_{p,q}\int_0^\infty x^r g(x,\zeta)(G(x,\zeta))^{2p+q+2}dx$$

Using (1.1) and (1.2) we get

$$\mu_r' = \sum_{p=0}^\infty \sum_{q=0}^\infty \delta_{p,q}\alpha\int_0^\infty x^{r+1}e^{-\frac{\alpha}{2}x^2}\left(1-e^{-\frac{\alpha}{2}x^2}\right)^{2p+q+2}dx$$

Using $(1-x)^a = \sum_{s=0}^\infty (-1)^s \binom{a}{s} x^a$

$$\mu_r' = \sum_{p,q,s=0}^\infty (-1)^s \delta_{p,q}\binom{2p+q+2}{s}\alpha\int_0^\infty x^{r+1}e^{-\frac{(s+1)\alpha}{2}x^2}dx$$

Making Substitution

$$\frac{(s+1)\alpha x^2}{2} = z \quad \Leftrightarrow \quad x = \sqrt{\frac{2z}{\alpha(s+1)}}$$

$$\mu_r' = \sum_{p,q,s=0}^{\infty} (-1)^s \delta_{p,q} \binom{2p+q+2}{s} \frac{\alpha}{2} \left(\frac{2}{\alpha(s+1)}\right)^{\frac{r+2}{2}} \int_0^{\infty} z^{\left(\frac{r+2}{2}\right)-1} e^{-z} dz$$

After solving the integral, we get

$$\mu_r' = \sum_{p,q,s=0}^{\infty} (-1)^s \delta_{p,q} \binom{2p+q+2}{s} \frac{\alpha}{2} \left(\frac{2}{\alpha(s+1)}\right)^{\frac{r+2}{2}} \Gamma\left(\frac{r+2}{2}\right)$$

4.1 Moment generating function

Moment generating function is given by

$$M_x(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} f(x,\alpha,\theta)dx$$

By using Taylor's expansion we get

$$M_x(t) = \int_0^{\infty} \left[1 + tx + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \dots\right] f(x,\alpha,\theta)dx$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^{\infty} x^r f(x,\alpha,\theta)dx$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} \sum_{p,q,s=0}^{\infty} (-1)^s \delta_{p,q} \binom{2p+q+4}{s} \frac{\alpha}{2} \left(\frac{2}{\alpha(s+1)}\right)^{\frac{r+2}{2}} \Gamma\left(\frac{r+2}{2}\right) \qquad \text{Using (2.1)}$$

$$M_x(t) = \sum_{p,q,r,s=0}^{\infty} (-1)^s \frac{t^r}{r!} \delta_{p,q} \binom{2p+q+4}{s} \frac{\alpha}{2} \left(\frac{2}{\alpha(s+1)}\right)^{\frac{r+2}{2}} \Gamma\left(\frac{r+2}{2}\right) \qquad (2.2)$$

The Characteristic function is obtained by replacing t by (it) in equation (2.2)

$$\Phi_x(t) = \sum_{p,q,r,s=0}^{\infty} (-1)^s \frac{(it)^r}{r!} \delta_{p,q} \binom{2p+q+4}{s} \frac{\alpha}{2} \left(\frac{2}{\alpha(s+1)}\right)^{\frac{r+2}{2}} \Gamma\left(\frac{r+2}{2}\right) \qquad (2.3)$$

5 Renyi Entropy

If X is a continuous random variable following Maxwell-Rayleigh distribution with pdf $f(x;\alpha,\theta)$, then

$$T_R(\delta)=\frac{1}{1-\delta}\log\left\{\int_0^\infty f^{\delta}(x,\varsigma)dx\right\}$$

$$=\frac{1}{1-\delta}\log\left\{\int_0^\infty\left(\sqrt{\frac{2}{\pi}}\,\frac{g(x,\theta,\varsigma)}{\theta}\,\frac{[G(x,\varsigma)]^2}{[\overline{G}(x,\varsigma)]^4}e^{-\frac{1}{2\theta^{32}}\left(\frac{G(x,\varsigma)}{\overline{G}(x,\varsigma)}\right)^2}\right)^{\delta}dx\right\}$$

$$=\frac{1}{1-\delta}\log\left\{\int_0^\infty\left(\left(\frac{1}{\theta}\sqrt{\frac{2}{\pi}}\right)^{\delta}[g(x,\theta,\varsigma)]^{\delta}\,\frac{[G(x,\varsigma)]^{2\delta}}{[\overline{G}(x,\varsigma)]^{4\delta}}e^{-\frac{\delta}{2\theta^{32}}\left(\frac{G(x,\varsigma)}{\overline{G}(x,\varsigma)}\right)^2}\right)dx\right\}$$

Use $e^{-mz}=\sum_{p=0}^{\infty}\frac{(-1)^p}{p!}(mz)^p$

$$T_R(\delta)=\frac{1}{1-\delta}\log\left\{\int_0^\infty\left(\left(\frac{1}{\theta}\sqrt{\frac{2}{\pi}}\right)^{\delta}[g(x,\theta,\varsigma)]^{\delta}\,\frac{[G(x,\varsigma)]^{2\delta}}{[\overline{G}(x,\varsigma)]^{4\delta}}\sum_{p=0}^{\infty}\frac{(-1)^p}{p!}\left(\frac{\delta}{2\theta^2}\right)^p\left(\frac{G(x,\zeta)}{\overline{G}(x,\zeta)}\right)^{2p}\right)dx\right\}$$

$$=\frac{1}{1-\delta}\log\left\{\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(-1)^p}{p!}\binom{4\delta+2p+q}{q}\left(\frac{\delta}{2\theta^2}\right)^p\left(\frac{1}{\theta}\sqrt{\frac{2}{\pi}}\right)^{\delta}\int_0^\infty\left([g(x,\theta,\varsigma)]^{\delta}(G(x,\zeta))^{2\delta+2p+q}\right)dx\right\}$$

$$=\frac{1}{1-\delta}\log\left\{\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\omega_{p,q}\int_0^\infty\left(\left[\alpha xe^{-\frac{\alpha}{2}x^2}\right]^{\delta}\left(1-e^{-\frac{\alpha}{2}x^2}\right)^{2\delta+2p+q}\right)dx\right\}$$

Where $\omega_{p,q}=\frac{(-1)^p}{p!}\binom{4\delta+2p+q}{q}\left(\frac{\delta}{2\theta^2}\right)^p\left(\frac{1}{\theta}\sqrt{\frac{2}{\pi}}\right)^{\delta}$

$$T_R(\delta)=\frac{1}{1-\delta}\log\left\{\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{s=0}^{\infty}(-1)^s\omega_{p,q}\binom{2\delta+2p+q}{s}\alpha^{\delta}\int_0^\infty x^{\delta}e^{-(s+\delta)\frac{\alpha}{2}x^2}dx\right\}$$

After making substitution $\frac{(s+\delta)\alpha x^2}{2}=z \quad\Leftrightarrow\quad x=\sqrt{\frac{2z}{\alpha(s+\delta)}}$, we get

$$T_R(\delta)=\frac{1}{1-\delta}\log\left\{\sum_{p,q,s=0}^{\infty}(-1)^s\omega_{p,q}\binom{2\delta+2p+q}{s}\frac{\alpha^{\delta}}{2}\left(\frac{2}{\alpha(\delta+s)}\right)^{\frac{\delta+1}{2}}\int_0^\infty z^{\frac{\delta+1}{2}-1}e^{-z}dz\right\}$$

After solving the integral, we have

$$T_R(\delta)=\frac{1}{1-\delta}\log\left\{\sum_{p,q,s=0}^{\infty}(-1)^s\omega_{p,q}\binom{2\delta+2p+q}{s}\frac{\alpha^{\delta}}{2}\left(\frac{2}{\alpha(\delta+s)}\right)^{\frac{\delta+1}{2}}\Gamma\left(\frac{\delta+1}{2}\right)\right\}$$

6 Tsallis Entropy

If X is a continuous random variable following Maxwell-Rayleigh distribution with pdf $f(x;\alpha,\theta)$, then

$$S_\delta = \frac{1}{\delta-1}\log\left\{1-\int_0^\infty f^\delta(x,\varsigma)dx\right\}$$

$$= \frac{1}{\delta-1}\log\left\{1-\int_0^\infty\left(\sqrt{\frac{2}{\pi}}\,\frac{g(x,\theta,\varsigma)}{\theta}\,\frac{[G(x,\varsigma)]^2}{[\bar{G}(x,\varsigma)]^4}e^{-\frac{1}{2\theta^{32}}\left(\frac{G(x,\varsigma)}{\bar{G}(x,\varsigma)}\right)^2}\right)^\delta dx\right\}$$

$$= \frac{1}{\delta-1}\log\left\{1-\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\omega_{p,q}\int_0^\infty\left(\left[\alpha x e^{-\frac{\alpha}{2}x^2}\right]^\delta\left(1-e^{-\frac{\alpha}{2}x^2}\right)^{2\delta+2p+q}\right)dx\right\}$$

$$= \frac{1}{\delta-1}\log\left\{1-\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{s=0}^{\infty}(-1)^s\omega_{p,q}\binom{2\delta+2p+q}{s}\alpha^\delta\int_0^\infty x^\delta e^{-(s+\delta)\frac{\alpha}{2}x^2}dx\right\}$$

After solving the integral, we get

$$S_\delta = \frac{1}{\delta-1}\log\left\{1-\sum_{p,q,s=0}^{\infty}(-1)^s\omega_{p,q}\binom{2\delta+2p+q}{s}\frac{\alpha^\delta}{2}\left(\frac{2}{\alpha(\delta+s)}\right)^{\frac{\delta+1}{2}}\Gamma\left(\frac{\delta+1}{2}\right)\right\}$$

7 Mean Residual Function

Mean residual function is given by:

$$m(x;\alpha,\theta) = \frac{1}{S_x(x)}\int_0^\infty t f(t,\alpha,\theta)dt - x$$

$$m(x;\alpha,\theta) = \frac{1}{S_x(x)}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{s=0}^{\infty}(-1)^s\delta_{p,q}\binom{2p+q+2}{s}\alpha\int^\infty t^2 e^{-\frac{(s+1)\alpha}{2}t^2}dt - x$$

After making substitution $\dfrac{(s+\delta)\alpha x^2}{2} = z \quad \Leftrightarrow \quad x = \sqrt{\dfrac{2z}{\alpha(s+\delta)}}$, we get

$$m(x;\alpha,\theta) = \frac{1}{S_x(x)}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{s=0}^{\infty}(-1)^s\delta_{p,q}\binom{2p+q+2}{s}\frac{\alpha}{2}\left(\frac{2}{\alpha(s+1)}\right)^{\frac{3}{2}}\int_{\frac{(s+1)\alpha x^2}{2}}^\infty z^{\frac{1}{2}}e^{-z}dz - x$$

After solving the integral, we get

$$m(x;\alpha,\theta)=\frac{1}{S_x(x)}\sum_{p,q,s=0}^{\infty}(-1)^s\delta_{p,q}\binom{2p+q+2}{s}\frac{\alpha}{2}\left(\frac{2}{\alpha(s+1)}\right)^{\frac{3}{2}}\gamma\left(\frac{3}{2},\frac{(s+1)\alpha x^2}{2}\right)-x$$

8 Incomplete Moments

Incomplete moments are given by:

We know that

$$T_q(s)=\int_0^s x^s f(x,\alpha,\theta)dx$$

$$=\int_0^s x^s\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(-1)^p}{p!}\binom{2p+q+3}{q}\sqrt{\frac{2}{\pi}}\frac{2}{(2\theta^2)^{p+1}}\alpha x e^{-\frac{\alpha}{2}x^2}\left(1-e^{-\frac{\alpha x^2}{2}}\right)^{2p+q+2}dx$$

$$=\int_0^s x^s\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(-1)^p}{p!}\binom{2p+q+3}{q}\sqrt{\frac{2}{\pi}}\frac{2}{(2\theta^2)^{p+1}}\alpha x e^{-\frac{\alpha}{2}x^2}\sum_{r=0}^{\infty}(-1)^r\binom{2p+q+2}{r}e^{-\left(\frac{\alpha x^2}{2}\right)^r}dx$$

$$=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}(-1)^r\kappa_{p,q}\int_0^s x^r e^{-\left(\frac{(r+1)\alpha x^2}{2}\right)}\alpha x dx$$

Where $\kappa_{p,q}=\frac{(-1)^p}{p!}\binom{2p+q+3}{q}\sqrt{\frac{2}{\pi}}\frac{2}{(2\theta^2)^{p+1}}\binom{2p+q+2}{r}$

Making substitution $\frac{(r+1)\alpha x^2}{2}=z \quad \Leftrightarrow \quad x=\sqrt{\frac{2z}{\alpha(r+1)}}$, we get

$$T_q(s)=\sum_{p,q,r=0}^{\infty}(-1)^r\kappa_{p,q}\frac{\alpha}{2}\left(\frac{2}{\alpha(r+1)}\right)^{\frac{r+2}{2}}\int_0^{\frac{(r+1)\alpha s^2}{2}} x^{\frac{r+2}{2}-1}e^{-z}dz$$

After solving the integral, we get

$$T_q(s)=\sum_{p,q,r=0}^{\infty}(-1)^r\kappa_{p,q}\frac{\alpha}{2}\left(\frac{2}{\alpha(r+1)}\right)^{\frac{r+2}{2}}\gamma\left(\frac{r+2}{2},\frac{(r+1)\alpha s^2}{2}\right)$$

9 Order Statistics

Let $x_{(1)},x_{(2)},x_{(3)},...,x_{(n)}$ denotes the order statistics of n random samples drawn from Maxwell- Rayleigh Distribution, then the pdf of $x_{(k)}$ is given by

$$f_{x(k)}(x;\alpha,\theta)=\frac{n!}{(k-1)!(n-k)!}f_X(x)\left[F_X(x)\right]^{k-1}\left[1-F_X(x)\right]^{n-k}$$

$$f_{x(k)}(x;\alpha,\theta)=\frac{n!}{(k-1)!(n-k)!}\sqrt{\frac{2}{\pi}}\frac{\alpha x}{\theta^2}e^{\frac{\alpha}{2}x^2}\left(e^{\frac{\alpha}{2}x^2}-1\right)^2e^{-\frac{1}{2\theta^2}\left[e^{\frac{\alpha}{2}x^2}-1\right]^2}$$

$$\times\left\{\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{2\theta^2}\left(e^{\frac{\theta}{2}x^2}-1\right)^2\right)\right\}^{k-1}\left(1-\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{2\theta^2}\left(e^{\frac{\theta}{2}x^2}-1\right)^2\right)\right)^{n-k}$$

Then the pdf of first order $X_{(1)}$ statistics of Maxwell-Rayleigh distribution is given by

$$f_{x(1)}(x;\alpha,\theta)=n\sqrt{\frac{2}{\pi}}\frac{\alpha x}{\theta^2}e^{\frac{\alpha}{2}x^2}\left(e^{\frac{\alpha}{2}x^2}-1\right)^2e^{-\frac{1}{2\theta^2}\left[e^{\frac{\alpha}{2}x^2}-1\right]^2}\left(1-\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{2\theta^2}\left(e^{\frac{\theta}{2}x^2}-1\right)^2\right)\right)^{n-1}$$

and the pdf of nth order $X_{(n)}$ statistics of Maxwell- Rayleigh Distribution is given by

$$f_{x(n)}(x;\alpha,\theta)=n\sqrt{\frac{2}{\pi}}\frac{\alpha x}{\theta^2}e^{\frac{\alpha}{2}x^2}\left(e^{\frac{\alpha}{2}x^2}-1\right)^2e^{-\frac{1}{2\theta^2}\left[e^{\frac{\alpha}{2}x^2}-1\right]^2}\left\{\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{2\theta^2}\left(e^{\frac{\theta}{2}x^2}-1\right)^2\right)\right\}^{n-1}$$

10 Maximum Likelihood Estimation

Let $x_1,x_2,x_3,\ldots,x_n$ be n random samples from Maxwell- Rayleigh Distribution, then its likelihood function is given by

$$l=\prod_{i=1}^{n}f(x;\alpha,\theta)$$
$$=\prod_{i=1}^{n}\left\{\sqrt{\frac{2}{\pi}}\frac{\alpha x}{\theta^2}e^{\frac{\alpha}{2}x^2}\left(e^{\frac{\alpha}{2}x^2}-1\right)^2e^{-\frac{1}{2\theta^2}\left[e^{\frac{\alpha}{2}x^2}-1\right]^2}\right\}$$

Its log likelihood function is given by

$$\log l=\log\left(\sqrt{\frac{2}{\pi}}\frac{\alpha}{\theta^2}\right)^n+\sum_{i=0}^{\infty}\log x_i+\sum_{i=0}^{\infty}\log\left(e^{\frac{\alpha}{2}x^2}-1\right)^2-\frac{\alpha}{2}\sum_{i=0}^{\infty}x_i-\frac{1}{2\theta^2}\sum_{i=0}^{\infty}\left(e^{\frac{\alpha}{2}x_i^2}-1\right)^2$$
$$=n\log\sqrt{\frac{2}{\pi}}+n\log\alpha-2n\log\theta+\sum_{i=0}^{\infty}\log x_i+2\sum_{i=0}^{\infty}\log\left(e^{\frac{\alpha}{2}x^2}-1\right)-\frac{\alpha}{2}\sum_{i=0}^{\infty}x_i-\frac{1}{\theta^2}\sum_{i=0}^{\infty}\left(e^{\frac{\alpha}{2}x_i^2}-1\right)$$

Differentiating w.r.t α and θ we will get

$$\frac{\partial \log l}{\partial \alpha}=\frac{n}{\alpha}+\sum_{i=0}^{\infty}\frac{x_i^{\,2}\,e^{\frac{\alpha}{2}x_i^{\,2}}}{\left(e^{\frac{\alpha}{2}x_i^{\,2}}-1\right)}.-\frac{1}{2}\sum_{i=0}^{\infty}x_i^{\,2}-\frac{1}{2\theta^2}\sum_{i=0}^{\infty}x_i^{\,2}\left(e^{\frac{\alpha}{2}x_i^{\,2}}-1\right)e^{\frac{\alpha}{2}x_i^{\,2}}$$

$$\frac{\partial \log l}{\partial \theta}=-\frac{2n}{\theta}+\frac{1}{\theta^3}\sum_{i-0}^{\infty}\left(e^{\frac{\alpha}{2}x_i^{\,2}}-1\right)^2$$

The above equations are non-linear equations and hence cannot be expressed in compact form. Therefore to solve these equations explicitly for α and θ is difficult. So we can apply iterative methods such as Newton–Raphson method, secant method, Regula-falsi method etc. The MLE of the parameters denoted as $\hat{\varsigma}\left(\hat{\alpha},\hat{\theta}\right)$of $\varsigma(\alpha,\theta)$can be obtained by using the above methods.

Since the MLE of $\hat{\varsigma}$ follows asymptotically normal distribution as given as follows

$\sqrt{n}(\hat{\varsigma}-\varsigma)\rightarrow N(0,I(\varsigma))$

Where $I^{-1}(\varsigma)$ is the limiting variance covariance matrix $\hat{\varsigma}$ and $I(\varsigma)$is a 2x2 Fisher Information matrix i.e

$$I(\varsigma)=\begin{bmatrix} E\left(\dfrac{\partial^2 \log l}{\partial \alpha^2}\right) & E\left(\dfrac{\partial^2 \log l}{\partial \alpha \partial \theta}\right) \\ E\left(\dfrac{\partial^2 \log l}{\partial \theta \partial \alpha}\right) & E\left(\dfrac{\partial^2 \log l}{\partial \theta^2}\right) \end{bmatrix}$$

Hence the approximate $100(1-\psi)$% confidence interval for α and θ are respectively given by

$$\hat{\alpha}\pm Z_{\frac{\psi}{2}}\sqrt{I_{\alpha\alpha}^{-1}(\hat{\varsigma})} \qquad \theta\pm Z_{\frac{\psi}{2}}\sqrt{I_{\theta\theta}^{-1}(\hat{\varsigma})}$$

Where $Z_{\frac{\psi}{2}}$ is the ψ^{th} percentile of the standard distribution.

11 Application

In this segment, the efficacy of the formulated distribution has been assessed using two real data sets. As the new distribution is compared to Exponentiated Exponential distribution (EED), Inverse Burr distribution (IBD), Rayleigh distribution (RD) and Maxwell distribution (MD). It is revealed from the tables that the new developed distribution offers an appropriate fit.

Various criterion including the AIC (Akaike information criterion), CAIC (Consistent Akaike information criterion), BIC (Bayesian information criterion) and HQIC (Hannan-Quinn information criteria) are used to compare the fitted models.

Data set 1: This data set contains observations on the strengths of 1.5 cm glass fibers, measured at the National Physical Laboratory, England. The observations are:

0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2 ,0.74,1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77,1.11, 1.28, 1.42, 1.5, 1.54, 1.6, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13,1.29, 1.48, 1.5, 1.55, 1.61, 1.62, 1.66, 1.7, 1.77, 1.84, 0.84, 1.24, 1.3, 1.48,1.51, 1.55, 1.61, 1.63.

Data set 2: This data set consists of 63 observations of the strengths of 1.5cm glass fibers by workers at the UK National Laboratory, reported by Smith and Naylor [11]. The data follows:

0.55, 0.74, 0.77, 0.81, 0.84, 0.93, 1.04, 1.11, 1.13, 1.24, 1.25, 1.27, 1.28, 1.29, 1.30, 1.36, 1.39, 1.42, 1.48, 1.48, 1.49, 1.49, 1.50, 1.50, 1.51, 1.52, 1.53, 1.54, 1.55, 1.55, 1.58, 1.59, 1.60, 1.61, 1.61, 1.61, 1.61, 1.62, 1.62, 1.63, 1.64, 1.66, 1.66, 1.66, 1.67, 1.68, 1.68, 1.69, 1.70, 1.70, 1.73, 1.76, 1.76, 1.77, 1.78, 1.81, 1.82, 1.84, 1.84, 1.89, 2.00, 2.01, 2.24

Descriptive Statistics for data set 1

Mean	Min.	Max.	Q_1	Q_3	Median	S.D	Skew.	Kurt.
1.49	0.55	2.24	1.33	1.67	1.55	0.32	-0.82	3.80

ML Estimates and standard error of the unknown parameters of data set 1

Model		MRD	EED	IBD	RD	MD
$\hat{\alpha}$		1.88457	2.62402	4.10252	0.86045	0.88021
$\hat{\theta}$		13.8979	30.4097	3.29700	---------	---------
	$\hat{\alpha}$	0.11690	0.24872	0.35106	0.11202	0.04678
S.E	$\hat{\theta}$	3.64755	9.53104	0.44333	---------	--------

Performance of distributions for data set 1

Model	MRD	EED	IBD	RD	MD
-2logl	-80.376	58.592	223.814	92.234	71.476
AIC	-76.376	62.953	115.907	94.234	73.4777
AICC	-76.162	62.807	116.121	94.304	73.547
HQIC	-74.754	62.215	117.529	95.045	74.288
BIC	-72.221	66.748	120.062	96.312	75.555

Descriptive Statistics for data set 2

Mean	Min.	Max.	Q_1	Q_3	Median	S.D	Skew.	Kurt.
1.507	0.550	2.240	1.375	1.685	1.590	0.324	-0.899	3.923

ML Estimates and standard error of the unknown parameters for data set 2

Model		MRD	EED	IBD	RD	MD
$\hat{\alpha}$		1.98429	2.61157	4.08864	0.84249	0.88955
$\hat{\theta}$		17.2263	31.3489	3.44170	---------	---------
	$\hat{\alpha}$	0.11651	0.23799	0.33566	0.10614	0.04575
S.E	$\hat{\theta}$	4.50125	9.51983	0.45049	---------	--------

Performance of distributions for data set 2

Model	MRD	EED	IBD	RD	MD
-2logl	-108.308	62.766	121.780	99.581	77.182
AIC	-104.308	66.766	125.780	101.581	79.182
AICC	-104.108	66.966	125.980	101.647	79.248
HQIC	-102.623	68.452	127.466	102.424	80.025
BIC	-100.022	71.053	130.067	103.724	81.325

As it is obvious from the above Performance of distribution tables for both data's that the newly formulated Maxwell- Rayleigh distribution has smaller values for AIC, AICC, BIC and HQIC as compared with the comparison models. Accordingly we arrive at the conclusion that Maxwell- Rayleigh distribution provides an adequate fit than the compared ones.

12 Conclusion

This newly introduced distribution "Maxwell-Rayleigh distribution" is obtained by T-X technique. Several mathematical properties for the newly formulated distribution are derived including moments, moment generating function, incomplete moments, order statistics etc. To show the behavior of p.d.f, c.d.f and other related measures different plots have been drawn. The parameters are obtained by MLE technique. Finally the efficiency of the explored distribution is examined through two real life engineering data sets which reveal that the explored distribution provides an adequate fit than compared ones.

Competing Interests

Authors have declared that no competing interests exist.

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Appendix

$$\frac{\partial^2 \log l}{\partial \alpha^2} = -\frac{n}{\alpha^2} - \frac{1}{2}\sum_{i=0}^{\infty} \frac{x_i^{\,4}}{\left(e^{\frac{\alpha}{2}x_i^{\,2}} - 1\right)} \cdot - \frac{1}{2\theta^2}\sum_{i=0}^{\infty} x_i^{\,4}\left(e^{\alpha x_i^{\,2}} - e^{\frac{\alpha}{2}x_i^{\,2}}\right)$$

$$\frac{\partial^2 \log l}{\partial \theta^2} = \frac{2n}{\theta^2} - \frac{3}{\theta^4}\sum_{i-0}^{\infty}\left(e^{\frac{\alpha}{2}x_i^{\,2}} - 1\right)^2$$

$$\frac{\partial^2 \log l}{\partial \alpha \partial \theta} = \frac{\partial^2 \log l}{\partial \theta \partial \alpha} = \frac{2}{\theta^3}\sum_{i-0}^{\infty}\left(e^{\frac{\alpha}{2}x_i^{\,2}} - 1\right)^2 e^{\alpha x_i^{\,2}}$$

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