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Research Article

Symmetry Analysis, Invariant Solutions, and Conservation Laws of Fractional KdV-Like Equation

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In this paper, Lie symmetries of time-fractional KdV-Like equation with Riemann-Liouville derivative are performed. With the aid of infinitesimal symmetries, the vector fields and symmetry reductions of the equation are constructed, respectively; as a result, the invariant solutions are acquired in one case; we show that the KdV-like equation can be reduced to a fractional ordinary differential equation (FODE) which is connected with the Erdélyi-Kober functional derivative; for this kind of reduced form, we use the power series method for extracting the explicit solutions in the form of power series solution. Finally, Ibragimov's theorem has been employed to construct the conservation laws.

1. Introduction

In recent times, more attention has been paid to fractional calculus. The research study of mathematical models including time-fractional order has been a significant title of many works in science and engineering areas. In fact, a physical process can rely not only on the instantaneous time but also on the history of time, which can be described via fractional calculus that renders the models described by fractional order most useful and practical than models represented by classical integer order. Fractional partial differential equations (FPDEs) have been progressively discussed by various researchers in different fields, such as viscoelasticity, vibration, biology, and fluid mechanics [1–6]. For the fractional derivative, there exists no unique notion to define its concept. There are several definitions of the fractional derivative, such as the Riemann-Liouville, the modified Riemann-Liouville, the Caputo, the Weyl, the Grunwald-Letnikov, and other derivatives that have been adopted by different researchers; for more details, see [2]. Recently, due to the various applications of FPDEs, resolving such equations and trying to find some analytic solutions become among the challenging problems, especially for nonlinear problems. Not long ago, many authors find solutions of FDEs, using several approaches including the sine-cosine method, homotopy perturbation method, first integral, Adomian decomposition method, and exp-function method; see [7–13]. In the end of the 19th century, Sophus Lie (1842-1899) introduced a new analytic method, called Lie group analysis, which is regarded as a powerful tool in the study of differential equation (DE) properties. After that, this approach has been developed by Ovsiannikov [14]. In general, Lie symmetry can be used for construction of the similarity solution, reducing order of equation, reducing the number of variables, acquiring the new solutions from the old one, and to provide linearisation of nonlinear PDEs. Furthermore, we use Lie symmetries to formulate the conservation laws (CLs) and many other applications; for more details, see [15–20].

One of the earliest studies was done by Gazizov et al. [21], by extending the Lie symmetry approach for FPDEs, and proposed prolongation formulae for fractional derivatives. Later, many researchers apply this approach for such type of time-fractional equation, particularly Riemann-Liouville derivative; see [22–28]. The popular work of Noether, known as Noether theorem [29], describes the linkage with symmetry of the Euler-Lagrange equation and conservation laws. Ibragimov [30] and Lukashchuk [31] have been successfully arrived to propose fractional generalization of Noether theorem and given the method to determine CLs of FDEs even if those equations are without fractional Lagrangian.

In this work, we use Lie symmetry analysis for nonlinear time-fractional KdV-like equation given by the following form:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{3}{2}\frac{u_x^3}{u^2} - \lambda u_{xxx} = 0, \tag{1}$$

where $\partial_t^\alpha u$ is the Riemman-Louivill (R-L) fractional derivative of order α with respect to t. If we put $\alpha = 1$, the fractional KdV-like equation becomes the classical nonlinear KdV-like equation, which was considered for extracting the solutions by using symmetry analysis (see [32]). The KdV-like equation is derived from the standard Korteweg-de Vries (KdV) differential equation

$$\frac{\partial u}{\partial t} + 6uu_x + u_{xxx} = 0, \tag{2}$$

which can be used to model a variety of different physical phenomena such as hydromagnetic collision-free waves, ion-acoustic waves, acoustic solitons in plasmas, lattice dynamics, stratified waves interior, and internal gravity waves (see, e.g., [33–36]). In [37], an inquiry was undertaken to increase the reliability and precision of a genetic programming-based method to deduce model equations from a proven analytical solution, especially by using the solitary wave solution; the program, instead of giving (2), surprisingly gave the fractional KdV-like equation. By using the KdV-like equations, we can find other properties of the classical equation (see [25, 32, 38–42]).

The paper is arranged as follows. In Section 2, we give some definitions and properties of fractional calculus. In Section 3, we present the Lie symmetry analysis for FPDEs. In Section 4, we compute the Lie point symmetries of fractional KdV-like equation, symmetry reductions, and some invariant solutions of Equation (1). In Section 5, we use the power series method for obtaining the explicit power series solutions. In Section 6, by using Ibragimov's theorem, the conservation laws of Equation (1) are obtained. Finally, we give some conclusions in Section 7.

2. Preliminaries of Fractional Calculus

The aim of this section is to provide some basic definitions and principle properties of fractional calculus that will be used during this study.

Definition 1 (see [2]). The Riemann-Liouville fractional integral of a real-valued function $f(x, t)$, of order $\alpha > 0$, with respect to variable t, is presented by

$$\left({}_0I_t^\alpha f\right)(x, t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(x, s)ds, t > 0, \tag{3}$$

where $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$ is the Euler gamma function.

Definition 2 (see [2]). The Riemann-Liouville fractional derivative of a real-valued function $f(x, t)$, of order $\alpha > 0$, with respect to variable t, is given by

$$\left({}_0D_t^\alpha f\right)(x, t) = \begin{cases} \dfrac{\partial^n}{\partial t^n} f(x, t), & \text{if } \alpha = n, \\ \dfrac{1}{\Gamma(n-\alpha)}\dfrac{\partial^n}{\partial t^n}\displaystyle\int_0^t (t-s)^{n-\alpha-1} f(x, s)ds, & 0 \le n-1 < \alpha < n, \end{cases} \tag{4}$$

where $n \in \mathbb{N}^*$.

Definition 3 (see [2]). The Leibnitz formula of two functions is given by the following expression:

$$D_t^\alpha(f(x, t)g(x, t)) = \sum_{n=0}^\infty \binom{\alpha}{n} D_t^{\alpha-n} f(x, t) D_t^n g(x, t), \tag{5}$$

where

$$\binom{\alpha}{n} = \frac{(-1)^{n-1}\alpha\Gamma(n-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)}. \tag{6}$$

Definition 4 (see [2]). The Erdélyi-Kober fractional differential operator of order α is defined by

$$\left(P_\lambda^{\delta,\alpha} g\right)(\zeta) = \prod_{i=0}^{m-1}\left(\delta + i - \frac{1}{\beta}\zeta\frac{d}{d\zeta}\right)\left(K_\lambda^{\delta+\alpha, m-\alpha} g\right)(\zeta),$$
$$m = \begin{cases} [\alpha] + 1, & \alpha \notin \mathbb{N}, \\ \alpha, & \alpha \in \mathbb{N}, \end{cases} \tag{7}$$

where

$$\left(K_\lambda^{\delta,\alpha} g\right)(\zeta) = \frac{1}{\Gamma(\alpha)}\int_1^\infty (z-1)^{\alpha-1} z^{-(\delta+\alpha)} g\left(\zeta z^{1/\lambda}\right)dz. \tag{8}$$

Lemma 5. *If we have*

$$t\frac{\partial}{\partial t}g(\varsigma) = \alpha\varsigma\frac{d}{d\varsigma}g(\varsigma), \textit{with}\, \varsigma = xt^{\alpha}, \tag{9}$$

then

$$\begin{aligned}&\frac{\partial^n}{\partial t^n}\left[t^{\chi}\left(K_{\lambda}^{\delta,n-\alpha}g\right)(\varsigma)\right]\\&= t^{\chi-n}\prod_{j=0}^{n-1}\left(\chi-n+1+j+\alpha\varsigma\frac{d}{d\varsigma}\right)\left(K_{\lambda}^{\chi-n+\alpha+1,n-\alpha}g\right)(\varsigma),\\&= t^{\chi-n}\left(P_{\lambda}^{\chi-n+1,\alpha}\right)(\varsigma).\end{aligned} \tag{10}$$

3. Principle Idea of Lie Symmetry Analysis for Time FPDEs

The main purpose of this section is to describe the fundamental concept of Lie symmetry approach applied for time FPDEs.

At first, let us consider the time FPDEs

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} = F(x,t,u,u_x,u_{xx},u_{xxx}), \tag{11}$$

where $\partial_t^{\alpha}u$ is R-L fractional derivative operator and subscripts indicate the partial derivatives.

Now, let us assume that Equation (11) is still invariant under the one parameter group of transformations presented by

$$\begin{aligned}\hat{t} &= t+\varepsilon\tau(x,t,u)+O(\varepsilon),\\ \hat{x} &= x+\varepsilon\xi(x,t,u)+O(\varepsilon),\\ \hat{u} &= u+\varepsilon\eta(x,t,u)+O(\varepsilon),\\ \hat{u}_{\hat{t}}^{\alpha} &= u_t^{\alpha}+\varepsilon\eta_t^{\alpha}(x,t,u)+O(\varepsilon),\\ \hat{u}_{\hat{x}} &= u_x+\varepsilon\eta_x^{(1)}(x,t,u)+O(\varepsilon),\\ \hat{u}_{\hat{x}\hat{x}} &= u_{xx}+\varepsilon\eta_{xx}^{(2)}(x,t,u)+O(\varepsilon),\\ \hat{u}_{\hat{x}\hat{x}\hat{x}} &= u_{xxx}+\varepsilon\eta_{xxx}^{(3)}(x,t,u)+O(\varepsilon),\end{aligned} \tag{12}$$

with ε being a group parameter, τ, ξ, and η present the infinitesimal functions, and $\eta_x^{(1)}$, $\eta_{xx}^{(2)}$, $\eta_{xxx}^{(3)}$, and η_t^{α} are the extended infinitesimals, which are presented as follows:

$$\begin{aligned}\eta_x^{(1)} &= D_x(\eta)-u_xD_x(\xi)-u_tD_x(\tau),\\ \eta_{xx}^{(2)} &= D_x\left(\eta_x^{(1)}\right)-u_{xx}D_x(\xi)-u_{xt}D_t(\tau),\\ \eta_{xxx}^{(3)} &= D_x\left(\eta_{xx}^{(2)}\right)-u_{xxx}D_x(\xi)-u_{xxt}D_x(\tau),\end{aligned} \tag{13}$$

where D_x is the total derivative operator with respect to x, introduced by

$$D_x = \frac{\partial}{\partial x}+u_x\frac{\partial}{\partial u}+u_{xx}\frac{\partial}{\partial u_x}+u_{tx}\frac{\partial}{\partial u_t}+\cdots, \tag{14}$$

and α-th order extended infinitesimal $\eta^{\alpha,t}$ is described by the following formula:

$$\begin{aligned}\eta_t^{\alpha} &= D_t^{\alpha}(\eta)+\xi D_t^{\alpha}(u_x)-D_t^{\alpha}(\xi u_x)\\&\quad+D_t^{\alpha}(D_t(\tau)u)-D_t^{\alpha-1}(\tau u)+\tau D_t^{\alpha+1}(u)\\&= \partial_t^{\alpha}(\eta)+(\eta_u-\alpha D_t(\tau))\partial_t^{\alpha}u-u\partial_t^{\alpha}(\eta_u)+\mu\\&\quad+\sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\partial_t^{n}\eta_u-\binom{\alpha}{n+1}D_t^{n+1}(\tau)\right]D_t^{\alpha-n}u\\&\quad-\sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^{n}(\xi)\partial_t^{\alpha-n}(u_x).\end{aligned} \tag{15}$$

Here, D_t^{α} refers to the total time-fractional derivative and μ is given by

$$\begin{aligned}\mu &= \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{r}{k}\frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}\\&\quad\times[-u]^r\frac{d^m}{dt^m}\left[u^{k-r}\right]\frac{\partial^{n-m+k}\eta_u}{\partial t^{n-m}\partial u^k}.\end{aligned} \tag{16}$$

The corresponding Lie algebra of symmetries (12) is spanned by the vector fields of the following form:

$$X = \tau(x,t,u)\frac{\partial}{\partial t}+\xi(x,t,u)\frac{\partial}{\partial x}+\eta(x,t,u)\frac{\partial}{\partial u}. \tag{17}$$

Theorem 6 (invariance criterion). *Equation (11) accepts X as an infinitesimal generator, if the Lie symmetry condition is satisfied, which can be presented as follows:*

$$X^{\alpha,3}(\partial_t^{\alpha}u-F)\big|_{(\partial_t^{\alpha}u-F)} = 0, \tag{18}$$

where the prolonged $X^{\alpha,3}$ operator of X takes the following form:

$$X^{\alpha,3} = X+\eta_t^{\alpha}\frac{\partial}{\partial_t^{\alpha}}+\eta_x^{(1)}\frac{\partial}{\partial u_x}+\eta_{xx}^{(2)}\frac{\partial}{\partial u_{xx}}+\eta_{xxx}^{(3)}\frac{\partial}{\partial u_{xxx}}. \tag{19}$$

Remark 7 (invariance condition). The transformation (12) must keep the lower limit of Equation (4) invariant; see [21]. So this condition is expressed as

$$\tau(x,t,u)\big|_{t=0} = 0. \tag{20}$$

Remark 8. We have to mention that the expression μ vanishes when the infinitesimal $\eta(x,t,u)$ is linear in the variable u that means

$$\eta(x,t,u) = u(x,t)g(x,t) + h(x,t). \tag{21}$$

Consequently, research was confined to the case when $\mu = 0$.

Definition 9. We can say that a solution $u = f(x,t)$ is an invariant solution of Equation (11) if we have the following conditions:

(i) $u = f(x,t)$ is an invariant surface of (12) that means

$$Xf = 0 \Rightarrow \left[\tau(x,t,,u)\frac{\partial}{\partial t} + \xi(x,t,u)\frac{\partial}{\partial x} + \eta(x,t,u)\frac{\partial}{\partial u}\right] f = 0. \tag{22}$$

(ii) $u = f(x,t)$ satisfies Equation (11)

So, we combine both conditions (i) and (ii); we find that a solution is invariant solution, if it satisfies the following characteristic function:

$$\tau u_t + \xi u_x = \eta, \tag{23}$$

which is equivalent to resolve the following equation:

$$\frac{dt}{\tau} = \frac{dx}{\xi} = \frac{du}{\eta}. \tag{24}$$

4. Lie Symmetries and Invariant Solutions for KdV-Like Equation

By using the results described in Section 2, we can determine the Lie symmetries, symmetry reductions, and invariant solutions of Equation (1).

4.1. Symmetries of KdV-Like Equation. Assuming that Equation (1) is an invariant under (12), we apply the third prolongation $X^{\alpha,3}$ to (1); we find the infinitesimal criterion (18) to be

$$\eta_t^\alpha - \frac{3u_x^3}{u^3}\eta + \frac{9}{2}\frac{u_x^2}{u^2}\eta_x^{(1)} - \lambda\eta_{xxx}^{(3)} = 0. \tag{25}$$

Substituting the explicit expressions $\eta, \eta_x^{(1)}, \eta_{xxx}^{(3)}$ and η_t^α into (25) and equating powers of derivatives up to zero, we get the determining of equations; solving this obtained determining system with the initial condition (20) shows that

$$\tau(x,t,u) = 3C_1 t, \xi(x,t,u) = \alpha C_1 x + C_2, \eta(x,t,u) = C_3 u, \tag{26}$$

with C_1, C_2, and C_3 as arbitrary constants. Now, we can achieve the 3-dimensional Lie algebra generated by the vector fields under the following formulas:

$$X_1 = \frac{\partial}{\partial x}, X_2 = 3t\frac{\partial}{\partial t} + \alpha x\frac{\partial}{\partial x}, X_3 = u\frac{\partial}{\partial u}. \tag{27}$$

Consequently, we get three Lie group of point transformations associated to Equation (1)

$$\begin{aligned} G_1 &: (x,t,u) \longrightarrow (x+\varepsilon, t, u), \\ G_2 &: (x,t,u) \longrightarrow \left(e^{\alpha\varepsilon}x, e^{3\varepsilon}t, u\right), \\ G_3 &: (x,t,u) \longrightarrow (x, t, e^{\varepsilon}u). \end{aligned} \tag{28}$$

Namely, if $f(x,t)$ is a solution of (1), then $g_i (i = 1,2,3)$ are also solutions of Equation (1)

$$\begin{aligned} g_1 &= f(x+\varepsilon, t), \\ g_2 &= f\left(e^{\alpha\varepsilon}x, e^{3\varepsilon}t\right), \\ g_3 &= e^{\varepsilon}f(x,t). \end{aligned} \tag{29}$$

4.2. The Similarity Reductions and Invariant Solutions of KdV-Like Equation. In this section, we perform the similarity reductions and reduced form of Equation (1), in order to obtain invariant solutions of KdV-like equation, from the corresponding vector field.

Case 1. Reduction with $X_1 = \partial/\partial x$, by integrating the characteristic equation

$$\frac{dt}{0} = \frac{dx}{1} = \frac{du}{0}, \tag{30}$$

we have the similarity variable $z = t$ and similarity function $f(z) = u$. Thus, we have

$$u_1 = f(t). \tag{31}$$

By replacing (31) into (1), Equation (1) is reduced to a FODE that has the following form:

$$D_t^\alpha f(t) = 0. \tag{32}$$

Consequently, the group-invariant solutions are expressed as the following form:

$$u_1 = k_1 t^{\alpha-1}, \tag{33}$$

where k_1 is an arbitrary constant.

Figure 1 presents the graph of solution $u_1(x,t)$ for some different values of α.

Case 2. Reduction with $X_{1,3} = aX_1 + bX_3$.

The similarity variable z and similarity function $f(z)$ of the generator $X_{1,3}$ are achieved by resolving the corresponding characteristic equation written as

$$\frac{dt}{0} = \frac{dx}{a} = \frac{du}{bu}. \tag{34}$$

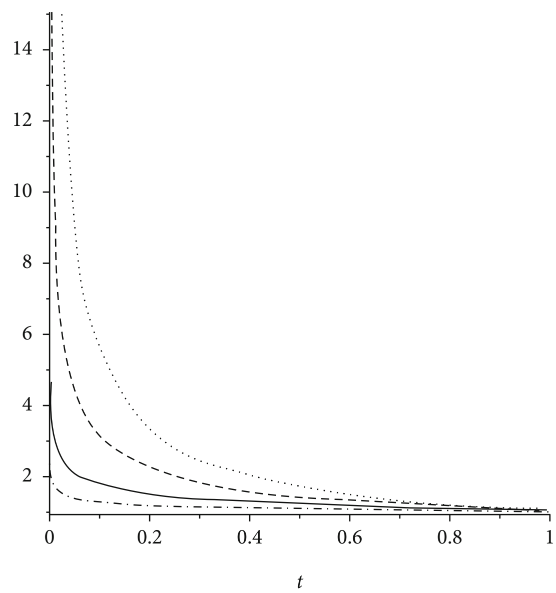


FIGURE 1: The solution $u_1(x,t)$ for Equation (1) for $k_1 = 1$.

So, we find

$$z = t, f(z) = e^{-b/ax}u. \quad (35)$$

Then, the corresponding invariant solution of Equation (1) has the form $u_2 = e^{(b/a)x}f(t)$; after substituting this function into Equation (1), we get the reduced equation in the form

$$D_t^\alpha f(t) + \left(\frac{3}{2} - \lambda\right)\left(\frac{b}{a}\right)^3 f(t) = 0. \quad (36)$$

Then,

$$f(t) = K_2 t^{\alpha-1} E_{\alpha,\alpha}\left[\left(\lambda - \frac{3}{2}\right)\left(\frac{b}{a}\right)^3 t^\alpha\right], \quad (37)$$

where $E_{\alpha,\beta}$ is the two-parametric Mittag-Leffer function given by

$$E_{\alpha,\beta}[z] = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k + \beta)}, \quad (38)$$

and K_2 is an arbitrary constant.

As a result, the explicit solution of Equation (1) has the following form:

$$u_2(x,t) = K_2 e^{b/ax} t^{\alpha-1} E_{\alpha,\alpha}\left[\left(\lambda - \frac{3}{2}\right)\left(\frac{b}{a}\right)^3 t^\alpha\right]. \quad (39)$$

Figure 2 presents the graph of solution $u_2(x,t)$ for some different values of α.

Case 3. Reduction with X_3, for this generator, we obtain a trivial solution.

Case 4. Reduction with X_2.

The similarity variable z and similarity transformation $f(z)$ accordding to X_2 are deduced from integrating the characteristic equation

$$\frac{dt}{3t} = \frac{dx}{\alpha x}. \quad (40)$$

We obtain

$$z = xt^{-\alpha/3}, f(z) = u. \quad (41)$$

Thus, the group-invariant solution has the form

$$u = f(z), z = xt^{-\alpha/3}. \quad (42)$$

Theorem 10. *If we consider the transformations (42), the fractional KdV-like equation is reduced to a nonlinear FODE of the form*

$$f^2\left(P_{3/\alpha}^{1-\alpha,\alpha} f\right)(z) + \frac{3}{2} f_z^3 - \lambda f^2 f_{zzz} = 0. \quad (43)$$

Proof. By using the Riemann-Liouville fractional derivative definition for the similarity transformation

$$u = f\left(xt^{-\alpha/3}\right), \quad (44)$$

we have

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1} f\left(xs^{-\alpha/3}\right) ds\right]. \quad (45)$$

Let $v = t/s$, we have $ds = (-t/v^2)dv$, so the above expression can be expressed as

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n}\left[t^{n-\alpha}\frac{1}{\Gamma(n-\alpha)}\int_1^\infty (v-1)^{n-\alpha-1} v^{-(n-\alpha+1)} f\left(zv^{\alpha/3}\right) dv\right], \quad (46)$$

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n}\left[t^{n-\alpha}\left(K_{3/\alpha}^{1,n-\alpha} f\right)(z)\right]. \quad (47)$$

On the other hand, we have

$$t\frac{\partial}{\partial t} f(z) = -tx\frac{\alpha}{3} t^{-\alpha/3-1} f'(z) = -\frac{\alpha}{3} z f'(z). \quad (48)$$

If $z = xt^{-\alpha/3}$, by using (10) in Lemma 5 for (46), we have

$$\frac{\partial^n}{\partial t^n}\left[t^{n-\alpha}\left(K_{3/\alpha}^{1,n-\alpha} f\right)(z)\right] = t^{-\alpha}\left(P_{3/\alpha}^{1-\alpha,\alpha} f\right)(z). \quad (49)$$

Continuing further by calculating u_x, u_{xx}, and u_{xxx} and substituting into Equation (1), the fractional KdV-like equation is reduced to following form:

$$f^2\left(P_{3/\alpha}^{1-\alpha,\alpha}f\right)(z)+\frac{3}{2}f_z^3-\lambda f^2 f_{zzz}=0. \tag{50}$$

So the proof becomes complete. □

5. The Power Series Solution of the Reduced Form of Fractional KdV-Like Equation

In this section, we employ the power series method, to explore the analytic solution of Equation (50); once we obtain the explicit solution of the reduced form, we can easily extract the power series solution of Equation (1).

We start by putting

$$f(z)=\sum_{n=0}^{\infty}a_n z^n, \tag{51}$$

which leads to

$$f_z=\sum_{n=0}^{\infty}n a_n z^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}z^n, \tag{52}$$

$$\begin{aligned} f_{zzz}&=\sum_{n=0}^{\infty}n(n-1)(n-2)a_n z^{n-3}\\ &=\sum_{n=0}^{\infty}(n+1)(n+2)(n+3)a_{n+3}z^n. \end{aligned} \tag{53}$$

Substituting (51), (52), and (53) into (50), we get the following formula:

$$\begin{aligned} &\left(\sum_{n=0}^{\infty}a_n z^n\right)^2\sum_{n=0}^{\infty}\frac{\Gamma(2-2\alpha+(n\alpha/3))}{\Gamma(2-\alpha+(n\alpha/3))}a_n z^n\\ &+\frac{3}{2}\left(\sum_{n=0}^{\infty}(n+1)a_{n+1}z^n\right)^3-\lambda\left(\sum_{n=0}^{\infty}a_n z^n\right)^2\\ &\cdot\left(\sum_{n=0}^{\infty}(n+3)(n+2)(n+1)a_{n+3}z^n\right)=0. \end{aligned} \tag{54}$$

When $n=0$, we get

$$a_3=\frac{1}{6\lambda}\left(\frac{\Gamma(2-2\alpha)}{\Gamma(2-\alpha)}a_0+\frac{3}{2}\frac{a_1^3}{a_0^2}\right). \tag{55}$$

Generally, for $n\ge 1$, we obtain

$$\begin{aligned} a_{n+3}=&\frac{1}{(n+1)(n+2)(n+3)\lambda a_0^2}\\ &\cdot\left[\sum_{k=0}^{n}\sum_{j=0}^{k}a_j a_{k-j}a_{n-k}\frac{\Gamma(2-2\alpha+((n-k)\alpha/3))}{\Gamma(2-\alpha+((n-k)\alpha/3))}\right.\\ &+\frac{3}{2}\sum_{j=0}^{n}\sum_{k=0}^{j}(j+1)(k-j+1)\\ &\left.\cdot(n-j+1)a_{k+1}a_{j-k+1}a_{n-j+1}\right]. \end{aligned} \tag{56}$$

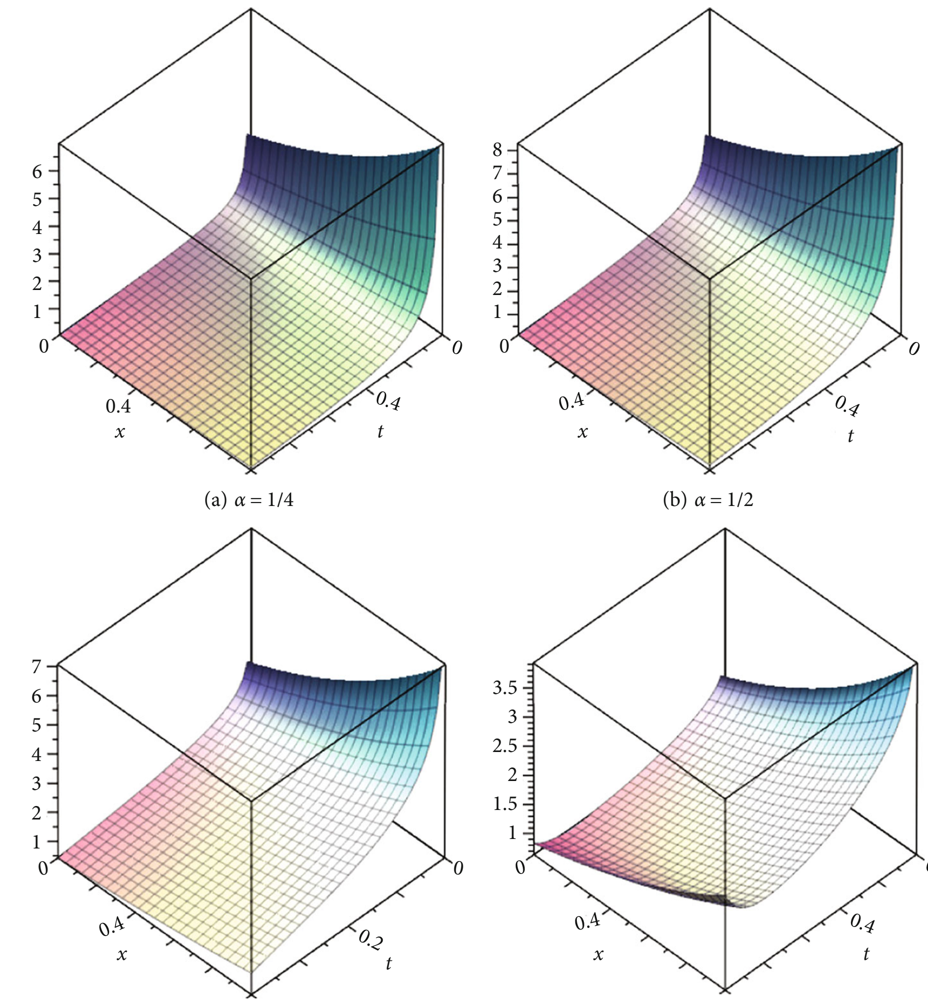
By using (56), all the coefficients a_n, $n\ge 3$ of power series solution are obtained by a systematic calculation and by choosing the suitable arbitrary constants a_1, a_2, and a_3 ($a_0\ne 0$).

Hence, the power series solution of Equation (50) is expressed as follows:

$$\begin{aligned} f(z)=&a_0+a_1z+a_2z^2+a_3z^3+\sum_{n=1}^{\infty}a_{n+3}z^{n+3}\\ =&a_0+a_1z+a_2z^2+\frac{1}{6\lambda}\left(\frac{\Gamma(2-2\alpha)}{\Gamma(2-\alpha)}a_0+\frac{3}{2}\frac{a_1^3}{a_0^2}\right)z^3\\ &+\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)(n+3)\lambda a_0^2}\\ &\cdot\left[\sum_{k=0}^{n}\sum_{j=0}^{k}a_j a_{k-j}a_{n-k}\frac{\Gamma(2-2\alpha+((n-k)\alpha/3))}{\Gamma(2-\alpha+((n-k)\alpha/3))}\right.\\ &+\frac{3}{2}\sum_{i=0}^{n}\sum_{j=0}^{i}(j+1)(k-j+1)\\ &\left.\cdot(n-k+1)a_{j+1}a_{k-j+1}a_{n-k+1}\right]z^{n+3}. \end{aligned} \tag{57}$$

Finally, the exact power series solution for Equation (1) is represented as

$$\begin{aligned} u_3(x,t)=&a_0+a_1xt^{-\alpha/3}+a_2x^2t^{-2\alpha/3}\\ &+\frac{1}{6\lambda}\left(\frac{\Gamma(2-2\alpha)}{\Gamma(2-\alpha)}a_0+\frac{3}{2}\frac{a_1^3}{a_0^2}\right)x^3t^{-\alpha}\\ &+\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)(n+3)\lambda a_0^2}\\ &\cdot\left[\sum_{k=0}^{n}\sum_{j=0}^{k}a_j a_{k-j}a_{n-k}\frac{\Gamma(2-2\alpha+((n-k)\alpha/3))}{\Gamma(2-\alpha+((n-k)\alpha/3))}\right.\\ &+\frac{3}{2}\sum_{i=0}^{n}\sum_{j=0}^{i}(j+1)(k-j+1)\\ &\left.\cdot(n-k+1)a_{j+1}a_{j-k+1}a_{n-k+1}\right]x^{n+3}t^{-n\alpha/3}. \end{aligned} \tag{58}$$



(a) $\alpha = 1/4$

(b) $\alpha = 1/2$

(c) $\alpha = 3/4$

(d) $\alpha = 9/10$

Figure 2: The solution $u_2(x,t)$ of Equation (1) for $K_2 = a = b = \lambda = 1$ and for $\alpha = 0.25, 0.5, 0, 75, 0.9$.

The graph of the power series solution is presented by Figures 3 and 4 for different values of α.

Case 1. For $\alpha = 0.25$.

Case 2. For $\alpha = 075$.

6. Conservation Laws of Fractional KdV-Like Equation

By using Ibragimov's method [30], we can obtain the conservation laws for time-fractional KdV-like equation.

First, we start by giving some definitions and results describing this method. Now, let us define the formal Lagrangian of fractional KdV-like equation

$$\mathfrak{L} = v\left(u_t^\alpha + \frac{3}{2}\frac{u_x^2}{u^2}u_x - \lambda u_{xxx}\right), \tag{59}$$

with $v(x,t)$ as a new dependent variable.

A vector $C = (C_t, C_x)$ that satisfies the following equation

$$D_t(C_t) + D_x(C_x) = 0, \tag{60}$$

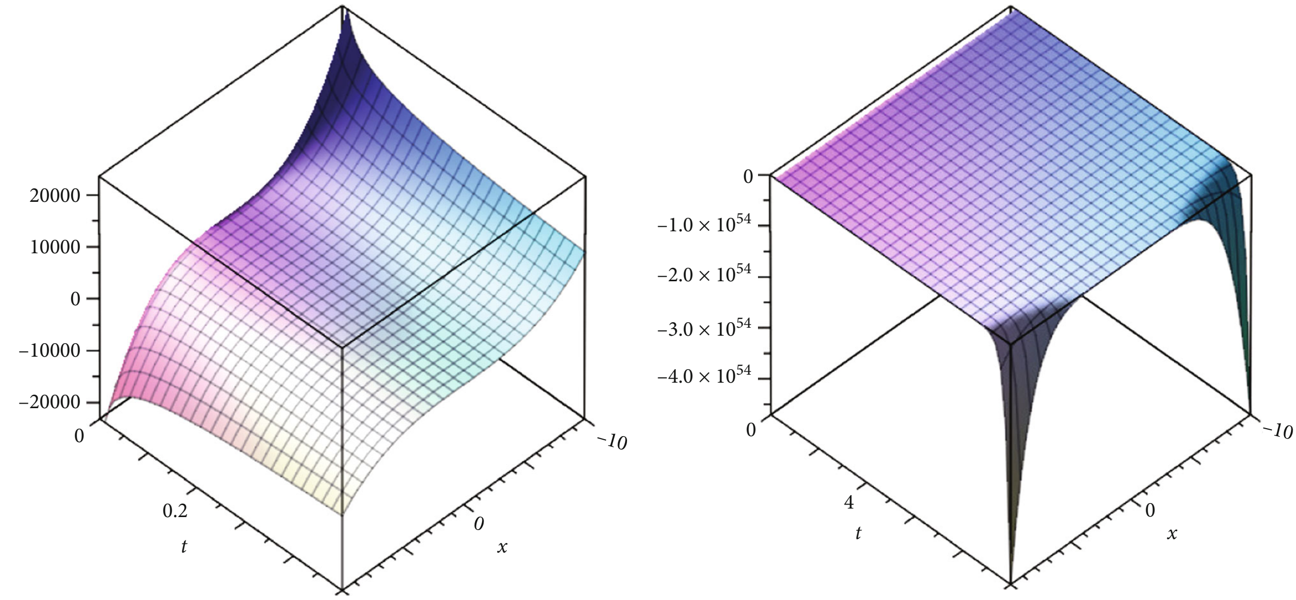
when $u(x,t)$ is a solution of (1), is named a conserved vector for time-fractional KdV-like equation, and Equation (60) is called the conservation equation.

The adjoint equation of KdV-like equation is expressed by the following formula:

$$\frac{\delta \mathfrak{L}}{\delta u} = 0, \tag{61}$$

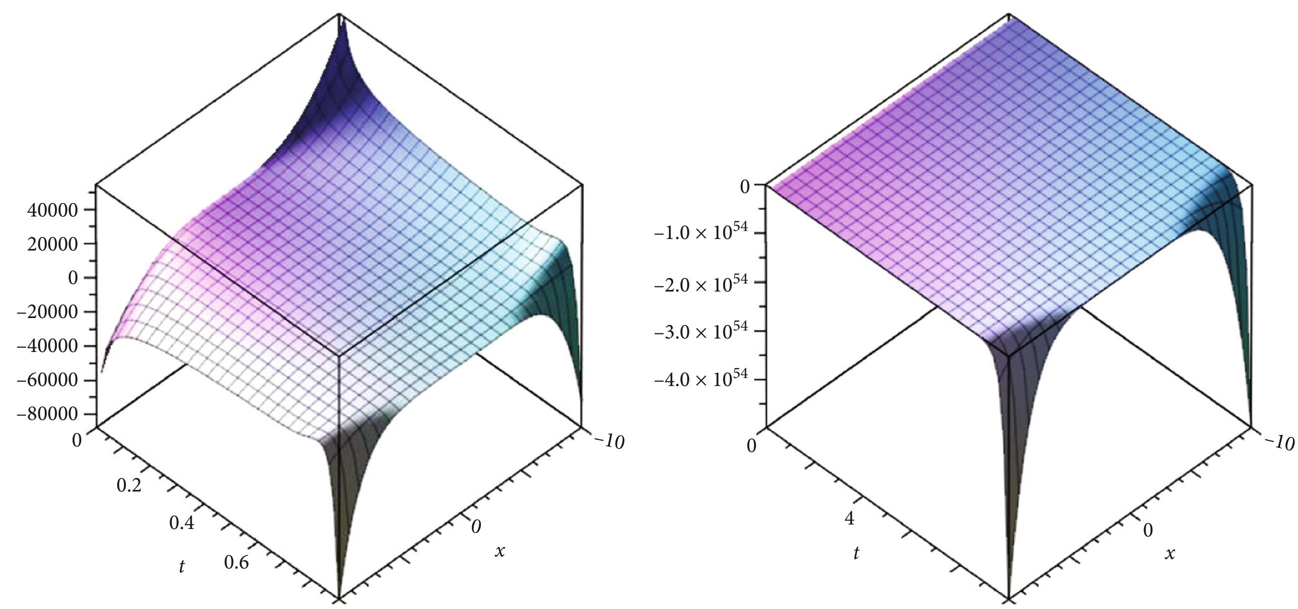
where $\delta/\delta u$ is the Euler-Lagrange operator which is presented as

$$\begin{aligned}\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + (D_t^\alpha)^* \frac{\partial}{\partial (D_t^\alpha u)} - D_x \frac{\partial}{\partial (u_x)} \\ + D_x^2 \frac{\partial}{\partial (u_{xx})} - D_x^3 \frac{\partial}{\partial (u_{xxx})} + \cdots,\end{aligned} \tag{62}$$



(a) $0 \le t \le 0.5$; $-10 \le x \le 10$ (b) $0 \le t \le 10$; $-10 \le x \le 10$

Figure 3: The solution $u_3(x,t)$ of Equation (1) for $a_0 = 2, a_1 = 5, a_2 = 1$.



(a) $0 \le t \le 0.9$; $-10 \le x \le 10$ (b) $0 \le t \le 10$; $-10 \le x \le 10$

Figure 4: The solution $u_3(x,t)$ of Equation (1) for $a_0 = 2, a_1 = 5, a_2 = 1$.

where $(D_t^\alpha)^*$ is the adjoint operator of D_t^α

$$(D_t^\alpha)^* = (-1)^n P_t^{n-\alpha}(D_t^n) = {}_cD_t^\alpha, \tag{63}$$

and $(D_t^\alpha)^*$ is the right-side Caputo operator, and

$$P_t^{m-\alpha} h(x,t) = \frac{1}{\Gamma(m-\alpha)} \int_t^s \frac{h(x,s)}{(s-t)^{1+\alpha-m}}\, ds. \tag{64}$$

The construction of CLs for FDEs is in the same way of PDEs, so we get

$$\Pr X_t^\alpha + D_t(\tau)\mathfrak{L} + D_x(\xi)\mathfrak{L} = W_i \frac{\delta \mathfrak{L}}{\delta u} + D_t(C_t) + D_x(C_x), \tag{65}$$

where W_i are the characteristic function given by

$$W_i = \eta_i - \tau_i u_t - \xi_i u_x. \tag{66}$$

On the other hand, for R-L time-fractional derivative, C_t is determined by

$$C_t = \sum_{j=0}^{m-1} (-1)^k D^{\alpha-1-k}(W_i) D_t^k \frac{\partial}{\partial D_t^\alpha u} - (-1)^m I\left(W_i, D_t^m \frac{\partial}{\partial (D_t^\alpha)} \right), \tag{67}$$

where I is defined by

$$I(f,h) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \int_t^T \frac{f(\tau,x)h(\phi,x)}{(\phi-\tau)^{\alpha+1-m}} d\phi d\tau. \tag{68}$$

So the components C_x and C_t are expressed by

$$\begin{aligned} C_t &= D_t^{\alpha-1}(W_i)\frac{\partial \mathfrak{L}}{\partial (D_t^\alpha u)} + I\left[W_i, D_t\left(\frac{\partial \mathfrak{L}}{\partial (D_t^\alpha u)} \right) \right], \\ C_x &= W_i \left[\frac{\partial \mathfrak{L}}{\partial u_x} - D_x\left(\frac{\partial \mathfrak{L}}{\partial u_{xx}} \right) + D_{xx}\left(\frac{\partial \mathfrak{L}}{\partial u_{xxx}} \right) \right] \\ &\quad + D_x(W_i)\left[\frac{\partial \mathfrak{L}}{\partial u_{xx}} - D_x\left(\frac{\partial \mathfrak{L}}{\partial u_{xxx}} \right) \right] + D_{xx}(W_i)\frac{\partial \mathfrak{L}}{\partial u_{xxx}}. \end{aligned} \tag{69}$$

By using the definition given above, we can compute the conservation laws of Equation (1).

As we have already seen previously, the time-fractional KdV-like equation admits three infinitesimal generators defined in Section 4:

$$X_1 = \frac{\partial}{\partial x}, X_2 = 3t\frac{\partial}{\partial t} + \alpha x \frac{\partial}{\partial x}, X_3 = u\frac{\partial}{\partial u}. \tag{70}$$

The characteristic functions corresponding to each generator are given by the following formulas:

$$W_1 = -u_x, W_2 = -3tu_t - \alpha x u_x, W_3 = u. \tag{71}$$

Substituting $W_i (i = 1, 2, 3)$ into the vector components (69), we obtain the conserved vectors of Equation (1)

$$\begin{aligned} C_t^1 &= vD_t^{\alpha-1}(-u_x) + I(-u_x, v_t), \\ C_x^1 &= -u_x\left[\frac{9}{2}\frac{u_x^2}{u^2} v + D_{xx}(-\lambda v) \right] \\ &\quad + D_x(-u_x)[D_x(\lambda v)] + D_{xx}(-u_x)[-\lambda v] \\ &= \lambda v_{xx} u_x - \frac{9}{2}\frac{u_x^3}{u^2} v - \lambda v_x u_{xx} + \lambda v u_{xxx}, \\ C_t^2 &= vD_t^{\alpha-1}(-3tu_t - \alpha x u_x) + I(-3tu_t - \alpha x u_x, v_t), \\ C_x^2 &= (-3tu_t - \alpha x u_x)\left[\frac{9}{2}\frac{u_x^2}{u^2} v + D_{xx}(-\lambda v) \right] \\ &\quad + D_x(-3tu_t - \alpha x u_x)[D_x(\lambda v)] \\ &\quad + D_{xx}(-3tu_t - \alpha x u_x)[-\lambda v] \\ &= (-3tu_t - \alpha x u_x)\left[\frac{9}{2}\frac{u_x^2}{u^2} v - \lambda v_{xx}) \right] \\ &\quad - (\alpha u_x + \alpha x u_{xx} + 3tu_{xt})[\lambda v_x] \\ &\quad + (2\alpha u_{xx} + \alpha x u_{xxx} + 3tu_{xxt})[\lambda v], \\ C_t^3 &= vD_t^{\alpha-1}(u) + I(u, v_t), \\ C_x^3 &= u\left[\frac{9}{2}\frac{u_x^2}{u^2} v + D_{xx}(-\lambda v) \right] + D_x(u)[D_x(\lambda v)] \\ &\quad + D_{xx}(u)[-\lambda v] = -\lambda v_{xx} u + \lambda v_x u_x \\ &\quad + \frac{9}{2}\frac{u_x^2}{u} v - \lambda v u_{xx}. \end{aligned} \tag{72}$$

7. Conclusion

In this work, we used symmetry analysis to study the time-fractional KdV-like equation. It has been shown that the KdV-like equation accepted three Lie point symmetries, which enable us to achieve symmetry reductions and construct new group invariant solutions of this equation; also, the equation is reduced to a nonlinear FODE where the fractional derivatives are in Erdélyi-Kober sense. Via power series method, another kind of solutions is obtained in the form of power series solutions. Finally, conservation laws were obtained for the KdV-like equation by applying Ibragimov's theorem.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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