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Multi-derivative Linear Multi-step Methods for Solving Third Order Boundary Value Problems

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Authors' contributions

This work was carried out in collaboration between both authors. Author EAA managed the literature searches and approved the draft manuscript. Author TVA wrote the first draft of the manuscript, managed the design and analyses of the study. Both authors read and approved the final manuscript.

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Abstract

Multi-derivative linear multi-step methods with continuous coefficients was derived through the block method approach using power series as basis function. Discrete scheme systems involving the multi-derivative linear methods were developed and their basic properties examined. The resulting schemes were used to solve general third order boundary value problems in ordinary differential equations without reducing it to first order. Numerical results were compared with the existing methods to show the accuracy and efficiency of the method. Results obtained show that our methods performed better than the existing methods.

Keywords: Third order; boundary value problems; multi-derivative; linear multi-step methods; Falkner-Skan.

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1 Introduction

Boundary value problems (BVPs) on third order differential equations have attracted a lot of attention in the literature in recent time. These kind of BVPs have many uses in the field of engineering and science, such as control theory, and biological sciences. In the recent time, tremendous attention has been shifted to developing methods for the solution of $y^{'''} = f(x, y, y', y'')$ subject to boundary conditions (see Awoyemi[1], Jator[2], Jator[3]). For instance, three-point fuzzy boundary value problems discussed in (Prakash[4]); laminar boundary layer and sandwich beam problems in (Sahi *et al.*[5]); numerical method for third order non linear BVP in engineering (Ikram[6]) are used in solving third order BVPs. Many of the methods above were solved by first reducing a higher order ordinary differential equation (ODE) to a lower order ODEs which takes a lot of human effort and computer time See Awoyemi[1].
This paper considers general third-order boundary value problems on the interval $\Delta \in [a, b]$

$$y^{'''} = f(x, y, y', y'') \tag{1.1}$$

subject to any of the boundary condition:
$y(a) = y_0, \quad y'(a) = \delta_0, \quad y(b) = y_N$
$y(a) = y_0, \quad y'(a) = \delta_0, \quad y'(b) = y_M$
$y(a) = y_0, \quad y'(b) = y_M, \quad y(c) = \gamma\frac{1}{2}$; where $c = (\frac{a+b}{2})$

where $y_0, \ \delta_0, \ \delta_1, \ \delta_2, \ y_N, \ y_M$ are constants and f is a continuous function that satisfies a lipschitz condition with respect to initial conditions as given by Sahi *et al.*[5].

The basic and auxiliary methods are obtained from the same continuous scheme and are of the same order, hence, possible errors which are due to auxiliary methods of lower order are avoided as the integration proceeds on the entire interval. The paper is organized as follows. Section 2 deals with how we derive an approximation $R(x)$ for $y(x)$ which is used to obtain the main and other multi-derivative linear multi-step methods (MDLMMs). Section 2 discussed how the methods were derived, while section 3 is deal with the analysis convergence of the method, computational aspects and an algorithm equipped with an automatic error estimate based on the double mesh principle. Numerical results are given in Section 4 to show speed and accuracy advantages. Summary and conclusion are in Section 5.

2 Construction of MDLMMs

The construction of the scheme (MDLMMs) is done here. Thus, on the interval $[x_n, \ x_{n+k}]$, the exact solution $y(x)$ and its derivatives are assumed to be locally represented by:

$$R_k(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + h^3 \sum_{j=0}^{k} \beta_j(x) f_{n+j} + h^4 \sum_{j=0}^{k} \omega_j(x) g_{n+j} + h^5 \sum_{j=0}^{k} \mu_j(x) \gamma_{n+j}, \tag{2.1}$$

$$\begin{cases} R_k^{'}(x) = \frac{d}{dx}\left(R(x)\right) \\ \\ R_k^{''}(x) = \frac{d^2}{dx^2}\left(R(x)\right) \end{cases} \tag{2.2}$$

where $\alpha_j(x), \ \beta_j(x), \ \omega_j(x), \ \mu_j(x)$ are continuous coefficients, and $m \geq 0$ is an integer. We assume that $y_{n+j} = R_k(x_n + jh)$ is the numerical approximation to the analytical solution $y(x_{n+j})$, $f_{n+j} = R'''{}_k(x_{n+jh})$, $g_{n+j} = R_k^{iv}(x_{n+jh})$ and $\gamma_{n+j} = R_k^{v}(x_{n+jh})$.

The continuous method (2) and its derivatives (3) are piecewise continuous on [a, b] and defined for all $x\epsilon[a,b]$. That is, $R_k(x)$, $R_k'''(x)$, $R_k^{iv}(x)$, $R_k^v(x)$ are defined such that $R_k(x) = y(x)+O(h^{12}))$, $R_k'(x) = \frac{d}{dx}(y(x) + O(h^{12}))$, $R_k''(x) = \frac{d^2}{dx^2}(y(x) + O(h^{12}))$, $x\epsilon(x_n, x_{n+3})$. The polynomials $R_0(x), R_3(x), ..., R_{N-3}(x)$, $R_0'''(x)$, $R_3'''(x), ..., R_{N-3}^{'''}(x), R_0^{iv}(x), R_3^{iv}(x), ..., R_{N-3}^{iv}(x), R_0^v(x), R_3^v(x)$, $..., R_{N-3}^v(x)$ then define piecewise polynomials $R(x)$, $R'(x)$, and $R''(x)$ which are also continuous on [a, b]. Hence, (2) and (3) have the ability to provide a continuous solution on [a, b] with a uniform accuracy comparable to that obtained at the grid points and can also be used to produce additional discrete methods (see Sahi *et al.*[5]). The following theorem as stated in (Sahi *et al.*[5]) facilitates the MDLMMs construction in (2) and (3). Thus

$$R_k(x_{n+j}) = y_{n+j}, j = 0, 1, 2. \tag{2.3}$$

$$R_k^{'''}(x_{n+j}) = f_{n+j}, \quad R_k^{iv}(x_{n+j}) = g_{n+j}, \quad R_k^v(x_{n+i}) = \gamma_{n+j} \quad j = 0, ..., 3. \tag{2.4}$$

the continuous representations (2) and (3) are equivalent to the following:

$$R_k(x) = V^T(S^{-1})^T H(x) \tag{2.5}$$

$$\begin{cases} R_k^{'}(x) = \frac{d}{dx}(V^T(S^{-1})^T H(x)) \\ R_k^{''}(x) = \frac{d^2}{dx^2}(V^T(S^{-1})^T H(x)) \end{cases} \tag{2.6}$$

where S is a matrix given as,

$$S = \begin{bmatrix} H_0(x_n) & H_1(x_n) & \cdots & H_{14}(x_n) \\ H_0(x_{n+1}) & H_1(x_{n+1}) & \cdots & H_{14}(x_{n+1}) \\ H_0(x_{n+2}) & H_1(x_{n+2}) & \cdots & H_{14}(x_{n+2}) \\ H_0^{'''}(x_n) & H_1^{'''}(x_n) & \cdots & H_{14}^{'''}(x_n) \\ H_0^{'''}(x_{n+1}) & H_1^{'''}(x_{n+1}) & \cdots & H_{14}^{'''}(x_{n+1}) \\ H_0^{'''}(x_{n+2}) & H_1^{'''}(x_{n+2}) & \cdots & H_{14}^{'''}(x_{n+2}) \\ H_0^{'''}(x_{n+3}) & H_1^{'''}(x_{n+3}) & \cdots & H_{14}^{'''}(x_{n+3}) \\ H_0^{iv}(x_n) & H_1^{iv}(x_n) & \cdots & H_{14}^{iv}(x_n) \\ H_0^{iv}(x_{n+1}) & H_0^{iv}(x_{n+1}) & \cdots & H_{14}^{iv}(x_{n+1}) \\ H_0^{iv}(x_{n+2}) & H_0^{iv}(x_{n+2}) & \cdots & H_{14}^{iv}(x_{n+2}) \\ H_0^{iv}(x_{n+3}) & H_0^{iv}(x_{n+3}) & \cdots & H_{14}^{iv}(x_{n+3}) \\ H_0^v(x_n) & H_1^v(x_n) & \cdots & H_{14}^v(x_n) \\ H_0^v(x_{n+1}) & H_0^v(x_{n+1}) & \cdots & H_{14}^v(x_{n+1}) \\ H_0^v(x_{n+2}) & H_0^v(x_{n+2}) & \cdots & H_{14}^v(x_{n+2}) \\ H_0^v(x_{n+3}) & H_0^v(x_{n+3}) & \cdots & H_{14}^v(x_{n+3}) \end{bmatrix}$$

$H_j(x_{n+j}) = x_{n+j}^j$,

$$V = [y_n, y_{n+1}, y_{n+2}, f_n, f_{n+1}, f_{n+2}, f_{n+3}, g_n, g_{n+1}, g_{n+2}, g_{n+3}, \gamma_n, \gamma_{n+1}, \gamma_{n+2}, \gamma_{n+3}]^T$$

$$J = [a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13}, a_{14}]^T$$

It should be noted that T denotes the transpose and $P_j(x) = x^j, j = 0, ..., 14$ are basis functions (See Sahi *et al.*[5]) for the proof. It is noted that the continuous methods (2) and (3) which are equivalent to the forms (6) and (7) are used to produce the main and additional methods which are combined and simultaneously applied to provide all approximations on the entire interval for boundary value problems of the form

$$y^{'''} = f(x, y, y', y'')$$

The continuous methods (2) and (3) are obtained by solving a system of 15 equations resulting from conditions (4) and (5) given in theorem mentioned above.

The coefficients of MDLMMs. To simplify the coefficients of (2), we introduce $q = \frac{x-x_{n+2}}{h}$; (see Sahi *et al.* [5]

Evaluating (2) at $x_{n+j},\ j = 3$, the main method is obtained by:

$$\begin{cases} y_3 - y_0 + 3\,y_1 - 3\,y_2 = \\ +h^3\left(\frac{7201}{14784}\,f_2 + \frac{7201}{14784}\,f_1 + \frac{191}{14784}\,f_3 + \frac{191}{14784}\,f_0\right) \\ +h^4\left(\frac{1613}{36960}\,g_1 + \frac{107}{41580}\,g_0 - \frac{107}{41580}\,g_3 - \frac{1613}{36960}\,g_2\right) \\ +h^5\left(\frac{233}{18480}\,\gamma_2 + \frac{17}{110880}\,\gamma_0 + \frac{233}{18480}\,\gamma_1 + \frac{17}{110880}\,\gamma_3\right) \end{cases} \tag{2.7}$$

The additional methods were obtained by evaluating (3) at $x_{n+j},\ j = 0, 1, 2, 3$.

$$\begin{cases} hy_0' + \frac{1}{2}\,y_2 - 2\,y_1 + \frac{3}{2}\,y_0 = \\ +h^3\left(\frac{55519}{864864}\,f_0 + \frac{31963}{104832}\,f_1 + \frac{1811}{494208}\,f_3 - \frac{11365}{288288}\,f_2\right) \\ +h^4\left(-\frac{74509}{5765760}\,g_1 + \frac{4057}{262080}\,g_2 - \frac{57697}{51891840}\,g_3 + \frac{298601}{25945920}\,g_0\right) \\ +h^5\left(-\frac{11443}{2882880}\,\gamma_2 + \frac{1637}{17297280}\,\gamma_3 + \frac{521}{786240}\,\gamma_0 + \frac{17393}{1153152}\,\gamma_1\right) \\ hy_1' + \frac{1}{2}\,y_0 - \frac{1}{2}\,y_2 = \\ +h^3\left(-\frac{295}{72072}\,f_2 - \frac{25525}{164736}\,f_1 - \frac{13439}{31135104}\,f_3 - \frac{14005}{1945944}\,f_0\right) \\ +h^4\left(-\frac{151}{120120}\,g_1 - \frac{25787}{17297280}\,g_0 + \frac{2153}{5765760}\,g_2 + \frac{3473}{25945920}\,g_3\right) \\ +h^5\left(\frac{323}{1153152}\,\gamma_2 - \frac{2383}{576576}\,\gamma_1 - \frac{4793}{51891840}\,\gamma_0 - \frac{151}{12972960}\,\gamma_3\right) \\ hy_2' - \frac{1}{2}\,y_0 + 2\,y_1 - \frac{3}{2}\,y_2 = \\ +h^3\left(\frac{25541}{288288}\,f_2 + \frac{276119}{1153152}\,f_1 - \frac{2087}{2830464}\,f_3 + \frac{6703}{1111968}\,f_0\right) \\ +h^4\left(\frac{123661}{5765760}\,g_1 + \frac{4273}{3706560}\,g_0 + \frac{107}{524160}\,g_3 - \frac{941}{45760}\,g_2\right) \\ +h^5\left(\frac{569}{262080}\,\gamma_2 + \frac{37963}{5765760}\,\gamma_1 - \frac{163}{10378368}\,\gamma_3 + \frac{241}{3706560}\,\gamma_0\right) \\ hy_3' - \frac{3}{2}\,y_0 + 4\,y_1 - \frac{5}{2}\,y_2 = \\ +h^3\left(\frac{597055}{576576}\,f_2 + \frac{797057}{1153152}\,f_1 + \frac{289117}{3459456}\,f_3 + \frac{39859}{1729728}\,f_0\right) \\ +h^4\left(\frac{72047}{1441440}\,g_1 + \frac{258001}{51891840}\,g_0 - \frac{398753}{25945920}\,g_3 - \frac{302933}{5765760}\,g_2\right) \\ +h^5\left(\frac{17819}{524160}\,\gamma_2 + \frac{43079}{2882880}\,\gamma_1 + \frac{193}{216216}\,\gamma_3 + \frac{1123}{3459456}\,\gamma_0\right) \end{cases}$$

$$\begin{cases} hy_0'' - y_2 + 2\,y_1 - y_0 = \\ +h^3\left(-\frac{4735777}{12972960}\,f_0 - \frac{1478063}{1921920}\,f_1 - \frac{651467}{51891840}\,f_3 + \frac{23489}{160160}\,f_2\right) \\ +h^4\left(\frac{186509}{5765760}\,g_1 - \frac{160499}{2882880}\,g_2 + \frac{65867}{17297280}\,g_3 - \frac{29069}{576576}\,g_0\right) \\ +h^5\left(\frac{39763}{2882880}\,\gamma_2 - \frac{1121}{3459456}\,\gamma_3 - \frac{22723}{8648640}\,\gamma_0 - \frac{262709}{5765760}\,\gamma_1\right) \\ \\ hy_1'' - y_2 + 2\,y_1 - y_n = \\ +h^3\left(-\frac{98447}{1921920}\,f_2 + \frac{31771}{960960}\,f_1 + \frac{171317}{77837760}\,f_3 + \frac{354953}{22239360}\,f_0\right) \\ +h^4\left(-\frac{114557}{2882880}\,g_1 + \frac{178049}{51891840}\,g_0 - \frac{5683}{8648640}\,g_3 + \frac{18925}{1153152}\,g_2\right) \\ +h^5\left(-\frac{17363}{5765760}\,\gamma_2 + \frac{1377}{320320}\,\gamma_1 + \frac{1427}{25945920}\,\gamma_3 + \frac{1637}{7413120}\,\gamma_0\right) \\ \\ hy_2'' - y_0 + 2\,y_1 - y_2 = \\ +h^3\left(\frac{218147}{480480}\,f_2 + \frac{344859}{640640}\,f_1 - \frac{473441}{155675520}\,f_3 + \frac{417149}{38918880}\,f_0\right) \\ +h^4\left(\frac{346253}{5765760}\,g_1 + \frac{16573}{8648640}\,g_0 + \frac{6359}{7413120}\,g_3 - \frac{240371}{2882880}\,g_2\right) \\ +h^5\left(\frac{1597}{192192}\,\gamma_2 + \frac{90059}{5765760}\,\gamma_1 - \frac{3503}{51891840}\,\gamma_3 + \frac{2551}{25945920}\,\gamma_0\right) \\ \\ h^2y_3'' - y_0 + 2\,y_1 - y_2 = \\ +h^3\left(\frac{804731}{640640}\,f_2 + \frac{46733}{137280}\,f_1 + \frac{9806759}{25945920}\,f_3 + \frac{1321877}{51891840}\,f_0\right) \\ +h^4\left(-\frac{991}{82368}\,g_1 + \frac{36793}{5765760}\,g_0 - \frac{458291}{8648640}\,g_3 - \frac{65119}{5765760}\,g_2\right) \\ +h^5\left(\frac{9583}{164736}\,\gamma_2 - \frac{683}{576576}\,\gamma_1 + \frac{24049}{8648640}\,\gamma_3 + \frac{8257}{17297280}\,\gamma_0\right) \end{cases} \tag{2.8}$$

3 Analysis of Convergence

The local truncation errors for the main methods of the (MDLMMs) is given by:

$$\tau_i = -\frac{1597}{28768836096000}\,y^{14}h^{14}\left(x_i+\theta_i\right)+O\left(h\right)^{15}, \quad h\tau_i' = -\frac{4079}{13277924352000}\,y^{14}h^{14}\left(x_i+\theta_i\right)h^{14}+O\left(h\right)^{15}$$

$$h^2\tau_i'' = -\frac{11881}{14384418048000}\,y^{14}h^{14}\left(x_i+\theta_i\right)+O\left(h\right)^{15}$$

$$i = 3, \cdots, N, \quad |\theta_i| \le 1$$

While that of the additional methods is given by:

$$\tau_1 = -\frac{251}{1120863744000}\,y^{14}h^{14}(\xi)+O\left(h\right)^{15}, \quad h\tau_1' = \frac{5557}{172613016576000}\,y^{14}h^{14}(\xi)+O\left(h\right)^{15}$$

$$h\tau_2' = \frac{383}{86306508288000}\,y^{14}h^{14}(\xi)+O\left(h\right)^{15}$$

$$\tau_2 = \frac{31}{40236134400}\,y^{14}h^{14}(\xi)+O\left(h\right)^{15}, \quad h^2\tau_1'' = -\frac{1577}{14384418048000}\,y^{14}h^{14}(\xi)+O\left(h\right)^{15}$$

$$h^2\tau_2'' = -\frac{173}{3196537344000}\,y^{14}h^{14}(\xi)+O\left(h\right)^{15}$$

$x_1 \le \xi \le x_2$

Convergence: Here, we show that the MDLMMs converged by compactly writing the main methods and additional methods in matrix form by introducing the following notations. Let D be a $3M \times 3M$ matrix defined by

$$D = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix}$$

In like manner, let U be a $3M \times 3M$ matrices given as

$$U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{bmatrix}$$

And;

$$\begin{aligned} C = (&hy_0' - \frac{3}{2}\,y_0 - \frac{55519}{864864}\,h^3 f_0 - \frac{298604}{25945920}\,h^4 g_0 - \frac{521}{786240}\,h^5\gamma_0, h^2 y_0'' - 2\,y_0 + \frac{4735777}{12972960}\,h^3 f_0 \\ &+ \frac{29069}{576576}\,h^4 g_0 - \frac{521}{786240}\,h^5\gamma_0, 0, \ldots 0, \\ &\frac{1}{2}\,y_0 + \frac{14005}{1945944}\,h^3 f_0 + \frac{25787}{17297280}\,h^4 g_0 + \frac{4793}{51891840}\,h^5\gamma_0, -y_0 - \frac{1780049}{22239360}\,h^3 f_0 \\ &- \frac{178049}{51891840}\,h^4 g_0 - \frac{1637}{7413120}\,h^5\gamma_0, 0, \ldots 0, \\ &- \frac{1}{2}\,y_0 - \frac{6703}{1111967}\,h^3 f_0 - \frac{4273}{3706560}\,h^4 g_0 - \frac{241}{3706560}\,h^5\gamma_0, -y_0 - \frac{417149}{38918880}\,h^3 f_0 \\ &- \frac{16573}{8648640}\,h^4 g_0 - \frac{2551}{25945920}\,h^5\gamma_0, 0, \ldots 0, \\ &- \frac{3}{2}\,y_0 - \frac{39859}{1729728}\,h^3 f_0 - \frac{258001}{51891840}\,h^4 g_0 - \frac{1123}{34594520}\,h^5\gamma_0, -y_0 - \frac{1321877}{51891840}\,h^3 f_0 \\ &- \frac{36793}{5765760}\,h^4 g_0 - \frac{8257}{17297280}\,h^5\gamma_0) \end{aligned}$$

We define the following vectors:

$$Y = \left(y(x_1), \ldots, y(x_N), hy'(x_1), \ldots, hy'(x_N), h^2 y''(x_1), \ldots, h^2 y''(x_N)\right)^T,$$

$$\overline{Y} = \left(y_1, \ldots, y_N, hy_1', \ldots, hy_N', h^2 y_1'', \ldots, h^2 y_N''\right)^T,$$

$$F = \left(f_1, \ldots, f_N, hg_1, \ldots, hg_N, h^2 k_1, \ldots, h^2 k_N, h^3 z_1, \ldots, h^3 z_N\right)^T$$

Remark: The variables k_i and z_i are introduced to augment the zero coefficients of matrix W

$$L(h) = (\tau_1, \ldots, \tau_N, h\tau_1' \ldots, h\tau_N', h^2\tau_1'', \ldots, \tau_N'')^T,$$

where $L(h)$ is the local truncation error.

$$E = \overline{Y} - Y = (e_1, \ldots, e_N, he_1' \ldots, he_N', h^2 e_1'', \ldots, e_N'')^T.$$

In the spirit of Jator [2] and Sahi *et al.*[5], we state the following necessary **Theorem** to justify the Order and Convergence of the Methods.

Accordingly, in agreement with **Theorem 3.1** in (Sahi *et al.*[5]), the multi-derivative linear multi-step Methods (MDLMMs) are Twelfth-Order Convergent Method. Thus, $\|E\| = O(h^{12})$ (See Sahi *et al.*[5]) for the proof.

Then, suppose $\|E\|$ is a norm of maximum error i.e. $\|E\| = max_i|\tau_i|$, and $E = (Q + M)^{-1}L(h)$, using the proof in Sahi *et al.*[5], it follows that $\|E\| = O(h^{12})$. Thus, MDLMMs is Twelfth-Order.

3.1 Computation

Here, a single matrix equation was formed from the main and additional methods of the multi-derivative linear multi-step methods which is used to solve (1.4) directly without reducing its order. We use a Mathematica 8.0 code, enhanced by the feature $NSolve[\ ]$ and $FindRoot[\ ]$ for linear and nonlinear problems. Mathematica 8.0 can symbolically compute derivatives and so the Jacobian matrix which involve the multi-derivatives are automatically generated. It then show how the multi-derivative linear multi-step methods (MDLMMs) is applied on the partition Γ_N, where

$$\Gamma_N := \{a = x_0 < x_1 < \ldots < x_N = b, \qquad x_n = x_0 + nh\}, \quad h = (b-a)/N$$

the discretization of problem (1.4) using the MDLMMs leads to $3N$ equations variables which are simultaneously solved while adjusting for the boundary conditions to yield the approximations $(y_n, y_n', y_n'')^T, \ \ n = 1, 2, 3, \ldots, N$.

4 Numerical Examples

Example 4.1: Consider the linear third order BVP that was solved by Sahi *et al.*[5].

$$y^{'''} - xy = (x^3 - 2x^2 - 5x - 3)e^x, \quad y(0) = y(1), y'(0) = 1, \ 0 \leq x \leq 1$$

$$Exact : y(x) = x(1-x)e^x$$

Table 1. Error for Example 4.1. Taking $h = 7$

$steps(N)$	MDLMMs Err	Sahi *et al.* Err[5]	Exact Solution
7	4.7335×10^{-20}	4.12×10^{-12}	0.433664
14	8.5718×10^{-24}	1.56×10^{-14}	0.433664
28	1.9430×10^{-27}	6.08×10^{-17}	0.429912
56	0.10×10^{-31}	2.37×10^{-19}	0.240755
112	0.10×10^{-31}	9.27×10^{-22}	0.106482

Remark: MDLMMs are compared with Fourth Derivative methods discussed in Sahi *et al.*[5] which solve the same problem for $h = 7$. It is observed that, the maximum errors 4.7335×10^{-20} obtained with MDLMMs is smaller than 4.12×10^{-12} of Sahi *et al.*[5]. Consequently, MDLMMs performed better in solution.

Example 4.2: We also consider another problem solved by Sahi *et al.*[5] (A nonlinear third order boundary-layer problem). We will consider the Falkner-Skan Equation ($\alpha = 1$)

$$y^{'''} + \alpha y y^{''} + \beta(1 - (y^{'})^2) = 0, \quad y(0) = y(0), y'(0) = 0, y'(\infty) = 1$$

There is no theoretical solution here.

Remark: Having considered different values of β (positive, zero and negative). The results as shown in the Table 4.2 shows that MDLMMs results are more efficient. The Number of steps needed in MDLMMs was only 7 to get the required values at the truncated boundary, whereas 10 steps was needed in Sahi *et al.*[5] and 21 in Brugano *et al.*[7]. Consequently, MDLMMs performed better in solution.

Table 2. Results for (Example 4.2) for different values of β

		MDLMMs	Sahi et al.[5]	BT[7]
$\eta\infty$	β	$y''(0)$	$y''(0)$	$y''(0)$
0.93	40	7.3144	7.314787	7.3149
0.95	30	6.33798	6.338219	6.33826
1.13	20	5.188058	5.180731	5.18076
1.49	10	3.6752	3.675257	3.67527
2.57	2	1.68732	1.687317	1.68732
2.88	1	1.23295	1.232951	1.23295
3.29	0.5	0.928234	0.928234	0.928234
4.01	0	0.471107	0.47110	0.471107
4.27	-0.1	0.321832	0.321838	0.321838
4.49	-0.15	0.22022	0.220244	0.220245
4.71	$-.18$	0.134875	0.134948	0.134948
5.00	-0.1988	0.0396817	0.039868	0.039859

Example 4.3: Here, we consider the following linear third-order differential equation solved by Awoyemi[1] using A P-stable linear multi-step method and Bhrawy *et al.*[8] using Jacobi-Gauss Collocation method.

$$y'''(q) - 2y''(q) - 3y'(q) + 10y(q) = 34qe^{-2q} - 16e^{-2q} - 10q^2 + 6q + 34, \quad q\epsilon[0, b]$$

$$\textit{Suject to the initial conditions}: y(0) = 3, \;\; y'(0) = 0, \;\; y''(0) = 0;$$

$$\textit{with the Exact solution}: y(q) = q^2e^{-2q} - q^2 + 3$$

Table 3. Error for Example 4.3. Taking $h = 10$

steps(N)	MDLMMs Err	Bhrawy *et al.* Err [8]	Exact Solution
10	1.3163×10^{-16}	1.18×10^{-6}	2.13534
20	2.3652×10^{-20}	3.92×10^{-16}	2.13534
30	1.4397×10^{-22}	3.73×10^{-16}	2.13534

Remark: Again as expected, error obtained with MDLMMs is smaller than that of Bhrawy *et al.*[8].Hence this methods show better accuracy compared with the existing method [8].

Example 4.4: We take another problem on non-linear third order BVP on: $0 \le x \le 1$, solved by Sahi *et al.*[5]

$$y''' = -2e^{-3y} + 4(1 + x)^{-3}; \;\; y(0) = 0, \;\; y'(0) = 1, \;\; y(1) = ln2;$$

$$\textit{with exact solution given as}: y(x) = ln(1 + x)$$

Table 4. Error (Err), Exact Solution and rate of convergence (RoC) for Example 4.4. Taking $h = 7$

steps(N)	MDLMMs Err	Sahi *et al.* Err[5]	Exact Solution	RoC
7	6.51175×10^{-13}	5.24×10^{-9}	0.451985	
14	1.68332×10^{-16}	2.39×10^{-11}	0.403465	11.918
28	3.1584×10^{-20}	9.50×10^{-14}	0.428996	12.380
56	6.0750×10^{-24}	3.62×10^{-16}	0.428996	12.344
112	1.2827×10^{-26}	2.27×10^{-17}	0.533775	12.209

Remark: MDLMMs is compared with Fourth Derivative methods discussed in Sahi *et al.*[5] which solve the same problem for $h = 7$. It is observed that, the maximum errors obtained with 3-CMDMs is smaller than that of Sahi *et al.*[5]. Hence, MDLMMs performed better.

5 Summary and Conclusion

A multi-derivative linear multi-step methods (MDLMMs) from which discrete Multi-derivative methods (MDMs) were obtained and applied for the solution of (1) subject to boundary conditions as stated without reducing to a lower order system was derived. Numerical results show the efficiency and accuracy advantages of the method over existing ones in the literature. The results are displayed in Tables 1-4. Thus, it is clear from the results given in the Tables that, the proposed multi-derivative linear multi-step Methods (MDLMMs) are very accurate compared with various existing methods.

Competing Interests

Authors have declared that no competing interests exist.

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