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A Study on the Solution of Linear Differential Equations with Polynomial Coefficients

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Authors' contributions

Authors KS and TM explained how the solutions of various linear differential equations with polynomial coefficients are obtained. Author TM summarized the methods in the form presented in this paper. Both authors collaborated to complete the manuscript. Both authors read and approved the final manuscript.

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Abstract

A linear differential equation with polynomial coefficients, which is expressed by $Lu(t) := \sum_{k=0}^{l_x} \sum_{m=0}^{m_x} a_{k,m} t^m \frac{d^k}{dt^k} u(t) = 0$ for $t > 0$, is studied, where $a_{k,m}$ are constants. In the present study, the lefthand side of the equation is rewritten as $Lu(t) := \sum_{l=-\infty}^{l_x} D_t^l u(t)$, where $D_t^l u(t) = \sum_{k=\max\{0,l\}}^{l_x} a_{k,k-l} t^{k-l} \frac{d^k}{dt^k} u(t)$, and each of $D_t^l u(t)$ is called a block of classified terms in $Lu(t)$. The solution is presented by taking advantage of the expression of the differential equation in terms of blocks of classified terms. When the differential equations is of the second order, six differential equations with two blocks of classified terms are chosen, such that their solutions are ordinarily expressed by the hypergeometric series, or the confluent hypergeometric series, or other two related series, except for some special values of coefficients. It is shown that all the other differential equations with two blocks of classified terms are reduced to one of these six by a change of variable.

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1 Introduction

In (Morita and Sato [1, 2, 3]) the solutions of Laplace's differential equation and of fractional differential equation of that type were discussed, where the differential equations are expressed by

$$(a_2t + b_2) \cdot {}_0D_R^{2\sigma}u(t) + (a_1t + b_1) \cdot {}_0D_R^{\sigma}u(t) + (a_0t + b_0)u(t) = f(t), \quad t > 0, \tag{1}$$

for $\sigma = 1$ and $\sigma = 1/2$. Here a_l, $b_l \in \mathbb{R}$ for $l \in \mathbb{Z}_{[0,2]}$ are constants, and ${}_0D_R^{l\sigma}u(t)$ are the Riemann-Liouville fractional derivatives [1], and their analytic continuations [2, 3]. Here $\mathbb{R}$ and $\mathbb{Z}$ are the sets of all real numbers and all integers, respectively, and $\mathbb{Z}_{[a,b]} = \{n \in \mathbb{Z} | a \leq n \leq b\}$ for $a, b \in \mathbb{Z}$ satisfying $a < b$. We also use $\mathbb{C}$ which is the set of all complex numbers, and $\mathbb{Z}_{>a} = \{n \in \mathbb{Z} | n > a\}$ $\mathbb{Z}_{<a} = \{n \in \mathbb{Z} | n < a\}$ for $a \in \mathbb{Z}$, and $\mathbb{R}_{>a} = \{x \in \mathbb{R} | x > a\}$ for $a \in \mathbb{R}$.

In the present paper, we study the differential equations of order $l_x \in \mathbb{Z}_{>0}$, with coefficients of polynomials, which are of the form:

$$\sum_{k=0}^{l_x} \sum_{m=0}^{\infty} a_{k,m} t^m \frac{d^k}{dt^k} u(t) = \sum_{k=0}^{l_x} (a_{k,0} + a_{k,1} \cdot t + a_{k,2} \cdot t^2 + a_{k,3} \cdot t^3 + \cdots) \cdot \frac{d^k}{dt^k} u(t) = 0, \quad t > 0, \tag{2}$$

where $a_{k,m}$ for $k \in \mathbb{Z}_{[0,l_x]}$ and $m \in \mathbb{Z}_{>-1}$ are constants. We assume that a finite number of the constants are nonzero.

In (Morita and Sato [2, 3, 4]), the solutions of special cases of Equation (1) or (2) were studied with the aid of distribution theory and the AC-Laplace transform, that is the Laplace transform supplemented by its analytic continuation. In the study, the following condition was adopted. Definition of the AC-Laplace transform and the formulas are given in 5.1, and used in Section 5.2.

Condition 1.1. $u(t)$ and $f(t)$ in (1) are expressed as a linear combination of $g_\nu(t) = \frac{1}{\Gamma(\nu)} t^{\nu-1}$ for $t > 0$ and $\nu \in S$, where S is a set of $\nu \in \mathbb{R}_{>-M} \backslash \mathbb{Z}_{<1}$ for some $M \in \mathbb{Z}_{>-1}$.

As a consequence, $u(t)$ is expressed as follows:

$$u(t) = \sum_{\nu \in S} u_{\nu-1} \frac{1}{\Gamma(\nu)} t^{\nu-1}, \tag{3}$$

where $u_{\nu-1} \in \mathbb{C}$ are constants. Because of this condition, obtained solutions are expressed by a power series of t multiplied by a power t^α:

$$u(t) = t^\alpha \sum_{k=0}^{\infty} p_k t^k, \tag{4}$$

where $\alpha \in \mathbb{C} \backslash \mathbb{Z}_{<0}$, $p_k \in \mathbb{C}$ and $p_0 \neq 0$.

A basic method of solving Equation (2) is to assume the solution in the form (4) with $\alpha \notin \mathbb{Z}_{<0}$. The solution is obtained by determining the coefficients p_k recursively; See e.g. Section 10.3 in Whittaker and Watson [5]. We present a formulation of this method, where we use $(z)_k$ and $(z)_k^-$ for $z \in \mathbb{C}$, $k \in \mathbb{Z}_{>-1}$, which denote $(z)_k = \prod_{m=0}^{k-1}(z+m)$ if $k \in \mathbb{Z}_{>0}$, and $(z)_0 = 1$, as usual, and

$$(z)_k^- = \prod_{m=0}^{k-1}(z-m) = (-1)^k(-z)_k, \quad k \in \mathbb{Z}_{>0}, \tag{5}$$

and $(z)_0^- = 1$.

We reassemble the terms of Equation (2) as

$$\sum_{l=-\infty}^{l_x} D_t^l u(t) = 0, \quad t > 0, \tag{6}$$

where

$$D_t^l u(t) = \sum_{k=\max\{0,l\}}^{l_x} a_{k,k-l} \cdot t^{k-l} \frac{d^k}{dt^k} u(t), \tag{7}$$

and call $D_t^l u(t)$ a block of classified terms. In fact, we confirm that the lefthand side of (2) is expressed as the lefthand side of (6), by writing $m = k - l$ in (2), as follows:

$$\sum_{k=0}^{l_x} \sum_{l=-\infty}^{k} a_{k,k-l} \cdot t^{k-l} \frac{d^k}{dt^k} u(t) = \sum_{l=-\infty}^{l_x} \sum_{k=\max\{0,l\}}^{l_x} a_{k,k-l} \cdot t^{k-l} \frac{d^k}{dt^k} u(t). \tag{8}$$

When $l_x = 2$, Equation (6) is expressed as

$$D_t^2 u(t) + D_t^1 u(t) + D_t^0 u(t) + D_t^{-1} u(t) + D_t^{-2} u(t) + \cdots = 0, \quad t > 0, \tag{9}$$

where

$$\begin{gathered} D_t^2 = a_{2,0} \frac{d^2}{dt^2}, \quad D_t^1 = a_{2,1} t \cdot \frac{d^2}{dt^2} + a_{1,0} \frac{d}{dt}, \quad D_t^0 = a_{2,2} t^2 \cdot \frac{d^2}{dt^2} + a_{1,1} t \cdot \frac{d}{dt} + a_{0,0}, \\ D_t^{-1} = a_{2,3} t^3 \cdot \frac{d^2}{dt^2} + a_{1,2} t^2 \cdot \frac{d}{dt} + a_{0,1} t, \quad D_t^{-2} = a_{2,4} t^4 \cdot \frac{d^2}{dt^2} + a_{1,3} t^3 \cdot \frac{d}{dt} + a_{0,2} t^2, \quad \cdots . \end{gathered} \tag{10}$$

When D_t^l is operated on t^α for $\alpha \in \mathbb{C} \backslash \mathbb{Z}$, we have

$$D_t^l t^\alpha = A_l(\alpha) t^{\alpha - l}, \tag{11}$$

where

$$A_l(\alpha) = \sum_{k=\max\{0,l\}}^{l_x} a_{k,k-l} \cdot (\alpha)_k^- . \tag{12}$$

For D_t^l given in (10), the $A_l(\alpha)$ satisfying (11) are given by

$$\begin{gathered} A_2(\alpha) = a_{2,0} \cdot (\alpha)_2^-, \quad A_1(\alpha) = a_{2,1} \cdot (\alpha)_2^- + a_{1,0}\alpha, \quad A_0(\alpha) = a_{2,2} \cdot (\alpha)_2^- + a_{1,1}\alpha + a_{0,0}, \\ A_{-1}(\alpha) = a_{2,3} \cdot (\alpha)_2^- + a_{1,2}\alpha + a_{0,1}, \quad A_{-2}(\alpha) = a_{2,4} \cdot (\alpha)_2^- + a_{1,3}\alpha + a_{0,2}, \quad \cdots . \end{gathered} \tag{13}$$

Theorem 1.1. Let $A_l(\alpha)$ be given by (12), and $\tilde{k} = \max\{k \in \mathbb{Z}_{[0,l_x]]} | a_{k,k-l} \neq 0\}$. Then $A_l(\alpha)$ is a polynomial of degree $\tilde{k}$. If $\tilde{k} > 0$, there exist roots of $A_l(\alpha) = 0$. Let $k_x \in \mathbb{Z}_{>0}$ be the total number of distinct roots of $A_l(\alpha) = 0$, which are α_k for $k \in \mathbb{Z}_{[0,k_x]}$. Then $A_l(\alpha)$ is expressed as

$$A_l(\alpha) = a_{\tilde{k},\tilde{k}-l} \prod_{k=1}^{k_x} (\alpha - \alpha_k)^{m_k}, \tag{14}$$

where $m_k \in \mathbb{Z}_{>0}$ for $k \in \mathbb{Z}_{[1,k_x]}$ satisfy $\sum_{k=1}^{k_x} m_k = \tilde{k}$. Then we have $\tilde{k}$ solutions of

$$D_t^l u(t) = 0, \tag{15}$$

which are given by t^{α_k}, and also

$$t^{\alpha_k} \log_e t, \cdots, t^{\alpha_k} (\log_e t)^{m_k - 1}, \tag{16}$$

if $m_k \geq 2$, for $k \in \mathbb{Z}_{[1,k_x]}$. If $\tilde{k} = 0$, there exists no solution of (15).

Remark 1.1. Equation (15) is often called Euler's differential equation, which is reduced to a differential equation with constant coefficients, that is $\prod_{k=1}^{k_x}(\frac{d}{dx}-\alpha_k)^{m_k}y(x)=0$, by the change of variable from t to $x=\log_e t$.

When we discuss a differential equation of order l_x, we adopt the following condition.

Condition 1.2. We consider such a differential equation of order l_x, that is not regarded as a differential equation of $u'(t)$, so that $\sum_{m=0}^{\infty}|a_{l_x,m}|\neq 0$ and $\sum_{m=0}^{\infty}|a_{0,m}|\neq 0$.

When only one nonzero block of classified terms exists in Equation (9), the following proposition follows from Theorem 1.1.

Proposition 1.1. *Let $A_l(\alpha)$ for $l\in\mathbb{Z}_{<3}$ be given by (13), and $\alpha=\alpha_1$ be a root of $A_l(\alpha)=0$. Then $u(t)=t^{\alpha_1}$ is a solution of $D_t^l u(t)=0$. If there exists another root α_2, we have another solution $u(t)=t^{\alpha_2}$. If not, but if $l=1$ and $a_{2,1}\neq 0$, or $l=0$ and $a_{2,2}\neq 0$, we have another solution given by $u(t)=t^{\alpha_1}\log_e t$.*

We now consider Equation (9) for the case in which there exist two or more nonzero blocks of classified terms and Condition 1.2 is satisfied. Let the first two nonzero blocks be $D_t^l u(t)$ and $D_t^{l-m_n}u(t)$, and the last nonzero block be $D_t^{l-m_x}u(t)$, so that $l\in\mathbb{Z}_{<3}$ and $m_n, m_x\in\mathbb{Z}_{>0}$ satisfy $m_n\leq m_x$. Then (9) is expressed as

$$(D_t^l+\sum_{m=m_n}^{m_x}D_t^{l-m})u(t)=0. \tag{17}$$

Remark 1.2. By (10) for $l_x=2$, we see that Equation (17) for $l=-1,-2,\ldots$ are equivalent to the one for $l=0$, and the differential equation for $l=1$ is equivalent to the one for $l=0$ when $a_{0,0}=0$. We note that the differential equation for $l=2$ is equivalent to a special one for $l=0$. Hence we study only the differential equation for $l=0$ in Section 2.2.

In Section 2, we seek those solutions of (17), that take the form of (4). In Section 2.1, we consider

$$D_t^l u(t)+D_t^{l-1}u(t)=0, \tag{18}$$

which is (17) for $m_n=m_x=1$, where the solutions in the form of (4) are given by the generalized hypergeometric function. In Section 2.2 and the following sections, discussion is restricted to the case of $l_x=2$. In Section 2.2, we consider Equation (18) for $l_x=2$ and $l=0$. There are six types of differential equation, whose solutions are expressed by

$${}_2F_1(a,b;c;z)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{k!(c)_k}z^k,\quad {}_2F_0(a,b;;z)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{k!}z^k, \tag{19}$$

$${}_1F_1(a;c;z)=\sum_{k=0}^{\infty}\frac{(a)_k}{k!(c)_k}z^k,\quad {}_0F_1(;c;z)=\sum_{k=0}^{\infty}\frac{1}{k!(c)_k}z^k. \tag{20}$$

The first series in (19) and (20) are the hypergeometric and the confluent hypergeometric series, respectively.

In Section 3, we consider

$$D_t^l u(t)+D_t^{l-m}u(t)=0,\quad m\in\mathbb{Z}_{>1}, \tag{21}$$

which is Equations (17) for $m_n=m_x=m>1$. There every differential equation under consideration is shown to be reduced to a differential equation of the form of (18) by a change of variable.

In Section 3.1, remarks are given on the parabolic cylinder function.

In some exceptional cases, the solution involving a logarithmic function appears. Comments are given on a method of obtaining a solution for such a case, in Section 4.

In Morita and Sato [3, 4], a part of results given in Section 2.2 are obtained by applying the AC-Laplace transform. In Section 5, an argument is given to show how the remaining results are obtained by the adopted method Morita and Sato [3, 4], where some preliminary formulas of the AC-Laplace transform are presented in Section 5.1 before the argument.

2 Basic Method of Solving Equation (17)

In this section, we seek the solution $u(t)$ of Equation (17), assuming that the solution is expressed by (4). In Section 4, remarks are given on solutions involving a logarithmic function.

When $u(t)$ is given by (4), we have

$$D_t^l u(t) = t^{\alpha - l} \sum_{k=0}^{\infty} A_l(\alpha + k) p_k t^k, \tag{22}$$

by using (11).

Substituting (4) in (17) and using (22), we obtain

$$\begin{aligned} &A_l(\alpha + k)p_k = 0, \quad k \in \mathbb{Z}_{[0, m_n - 1]}, \\ &A_l(\alpha + k)p_k + \sum_{m=m_n}^{m_x} A_{l-m}(\alpha + k - m)p_{k-m} = 0, \quad k = \mathbb{Z}_{>m_n - 1}, \end{aligned} \tag{23}$$

where we put $p_k = 0$ for $k \in \mathbb{Z}_{<0}$.

Definition 2.1. Let α satisfy $A_l(\alpha) = 0$, $p_0 \neq 0$, p_k for $k \in \mathbb{Z}_{>m_n-1}$ satisfy (23), and $p_k = 0$ for $k \in \mathbb{Z}_{[1, m_n-1]}$, if $m_n \in \mathbb{Z}_{>1}$. Then we denote $u(t)$ given by (4), by $\phi_\alpha(t)$.

We note that since $p_0 \neq 0$, $A_l(\alpha) = 0$ is required, and hence adopt the following condition.

Condition 2.1. There exists α satisfying $A_l(\alpha) = 0$.

Lemma 2.1. *If $A_l(\alpha)$ is a polynomial of α of degree 0 or 1, we have no or one solution, accordingly, of the form (4). When $l_x = 2$, if $l = 1$, $a_{2,1} = 0$ and $a_{1,0} \neq 0$, or $l = 0$, $a_{2,2} = 0$ and $a_{1,1} \neq 0$, then we have only one solution $\phi_0(t)$, the point $t = 0$ is called an irregular point of the differential equation, in Section 10.3 of Whittaker and Watson [5].*

Lemma 2.2. *When $l_x = 2$, if $l = 2$, or $l = 1$ and $a_{2,1} \neq 0$, or $l = 0$ and $a_{2,2} \neq 0$, the point $t = 0$ is called a regular point of the differential equation, in Section 10.3 of Whittaker and Watson [5]. Then we have two solutions of (17), among which those in the form (4) are as follows:*

1. *If $A_l(\alpha) = 0$ has two distinct roots, we call them α_1^* and α_2^*, so that $\text{Re}\,\alpha_1^* \geq \text{Re}\,\alpha_2^*$. Then one of the solutions is $\phi_{\alpha_1^*}(t)$. If $\alpha_1^* - \alpha_2^* \notin \mathbb{Z}$, the second one is $\phi_{\alpha_2^*}(t)$.*
2. *If $A_l(\alpha) = 0$ has only one root, which we call α_1. Then we have a solution $\phi_{\alpha_1}(t)$.*

Remark 2.1. Lemmas 2.1 and 2.2 show that α in the solutions of the form (4) are determined by the first nonzero block of classified terms in (6) or (9). In many studies and in the following part of the present paper, solutions are given for Equation (9) with two nonzero blocks. When we solve a differential equation with more than two nonzero blocks in (9), knowledge of the solutions of the differential equation which consists of the first two of the nonzero blocks is helpful.

When Equation (9) consists of two nonzero blocks of classified terms, and hence it is expressed as (17) with $m_n = m_x$, the recursion formula (23) is simple. Many differential equations which appear in mathematical physics are of this type.

2.1 Solution of equation (18)

We now study the solutions of Equation (18), which is (17) for $m_n = m_x = 1$.

We assume that D_t^l and D_t^{l-1} in (18) are differential operators of order $q+1 \in \mathbb{Z}_{>0}$ and $p \in \mathbb{Z}_{>-1}$, respectively, and hence $A_l(\alpha)$ and $A_{l-1}(\alpha)$ defined by (11) are polynomials of degree $q+1$ and p, respectively. We express these as

$$\begin{aligned} A_l(\alpha) &= \mu \prod_{n=1}^{q+1} (\alpha - \alpha_n), \\ A_{l-1}(\alpha) &= \nu \prod_{n=1}^{p} (\alpha - \beta_n), \quad p \in \mathbb{Z}_{>0}, \end{aligned} \tag{24}$$

and $A_{l-1}(\alpha) = \nu$ if $p = 0$, where $\mu \in \mathbb{R}$, $\nu \in \mathbb{R}$, $\alpha_n \in \mathbb{C}$ for $n \in \mathbb{Z}_{[1,q+1]}$, are constants, and $\beta_n \in \mathbb{C}$ for $n \in \mathbb{Z}_{[1,p]}$ are constants if $p \in \mathbb{Z}_{>0}$.

We show that one or more of the solutions are expressed by the generalized hypergeometric series given by

$$ {}_pF_q(a_1, \cdots, a_p; c_1, \cdots, c_q; z) = \sum_{k=0}^{\infty} \frac{\prod_{m=1}^{p} (a_m)_k}{k! \prod_{m=1}^{q} (c_m)_k} z^k, \quad \text{if } p \in \mathbb{Z}_{>0} \text{ and } q \in \mathbb{Z}_{>0}, \tag{25}$$

$$ {}_0F_q(; c_1, \cdots, c_q; z) = \sum_{k=0}^{\infty} \frac{1}{k! \prod_{m=1}^{q} (c_m)_k} z^k, \quad \text{if } p = 0 \text{ and } q \in \mathbb{Z}_{>0}, \tag{26}$$

$$ {}_pF_0(a_1, \cdots, a_p; ; z) = \sum_{k=0}^{\infty} \frac{\prod_{m=1}^{p} (a_m)_k}{k!} z^k, \quad \text{if } p \in \mathbb{Z}_{>0} \text{ and } q = 0. \tag{27}$$

Theorem 2.1. Let D_t^l and D_t^{l-1} in (18) be differential operators of order $q+1 \in \mathbb{Z}_{>0}$ and $p \in \mathbb{Z}_{>-1}$, respectively, and $A_l(\alpha)$ and $A_{l-1}(\alpha)$ defined by (11) be expressed as (24). If $n \in \mathbb{Z}_{[1,q+1]}$ is such that there exists no $m \in \mathbb{Z}_{[1,q+1]}$ for which $\alpha_m - \alpha_n \in \mathbb{Z}_{>0}$, then a solution of (18) is given by

$$\phi_{\alpha_n}(t) = \left\{ \begin{array}{ll} t^{\alpha_n} {}_pF_q(a_1, \cdots, a_p; c_1, \cdots, c_q; -\frac{\nu}{\mu} t), & \text{if } p \in \mathbb{Z}_{>0} \text{ and } q \in \mathbb{Z}_{>0}, \\ t^{\alpha_n} {}_0F_q(; c_1, \cdots, c_q; -\frac{\nu}{\mu} t), & \text{if } p = 0 \text{ and } q \in \mathbb{Z}_{>0}, \\ t^{\alpha_n} {}_pF_0(a_1, \cdots, a_p; ; -\frac{\nu}{\mu} t), & \text{if } p \in \mathbb{Z}_{>0} \text{ and } q = 0, \end{array} \right. \tag{28}$$

where $a_m = \alpha_n - \beta_m$ for $m \in \mathbb{Z}_{[1,p]}$, and c_m for $m \in \mathbb{Z}_{[1,q]}$ are given by $c_m = \alpha_m - \alpha_n + 1$ if $m < n$ and $c_m = \alpha_{m+1} - \alpha_n + 1$ if $m \geq n$.

Proof. Substituting (4) in (18) and using (22), we obtain $A_l(\alpha) = 0$, and

$$A_l(\alpha + k) p_k + A_{l-1}(\alpha + k - 1) p_{k-1} = 0, \tag{29}$$

for $k = \mathbb{Z}_{>0}$, in place of (23). We then obtain the solution (28) of (18), by choosing $\alpha = \alpha_n$ and determining p_k by (29).

Corollary 2.1. *If α_n for $n \in \mathbb{Z}_{[1,q+1]}$ are distinct with each other, and satisfy $\alpha_n - \alpha_m \notin \mathbb{Z}$ for every pair $n, m \in \mathbb{Z}_{[1,q+1]}$, then we have $q+1$ solutions of* (18)*, which are given by* (28)*.*

2.2 Solution of equation (18) for $l_x = 2$ and $l = 0$

From now on, we restrict the discussion to the case of $l_x = 2$.

We introduce notation ${}_n\tilde{D}_t^l$ which represents D_t^l when the coefficient of t^n is nonzero and those of t^m for $m > n$ are all zero. The differential equations belonging to Equation (18) for $l = 0$ are classified into

$$ {}_2\tilde{D}_t^0 u(t) + {}_n\tilde{D}_t^{-1} u(t) = 0, \quad n = 3, 2, 1, \tag{30}$$

$$ {}_1\tilde{D}_t^0 u(t) + {}_3\tilde{D}_t^{-1} u(t) = 0. \tag{31}$$

We call Equation (30) for $n = 3, 2$ and 1 as (30-3), (30-2) and (30-1), respectively.

We use a, b and c, which satisfy $a_{1,1} = a_{2,2}(1 + a + b)$ and $a_{0,0} = a_{2,2} \cdot ab$ when $a_{2,2} \neq 0$, and $a_{0,0} = a_{1,1} \cdot c$ when $a_{2,2} = 0$ and $a_{1,1} \neq 0$. Using these in (10), we obtain

$$ {}_2\tilde{D}_t^0 = a_{2,2}[t^2 \cdot \frac{d^2}{dt^2} + (1 + a + b)t \cdot \frac{d}{dt} + ab], \quad {}_1\tilde{D}_t^0 = a_{1,1}(t \cdot \frac{d}{dt} + c). \tag{32}$$

When $a_{0,0} = 0$, we put $b = 0$ and $c = 0$ in (32). We use $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$, which satisfy $a_{1,2} = a_{2,3}(1+\tilde{a}+\tilde{b})$ and $a_{0,1} = a_{2,3} \cdot \tilde{a}\tilde{b}$ when $a_{2,3} \neq 0$, and $a_{0,1} = a_{1,2} \cdot \tilde{c}$ when $a_{2,3} = 0$ and $a_{1,2} \neq 0$. Using these in (10), we obtain

$$ {}_3\tilde{D}_t^{-1} = a_{2,3} \cdot t[t^2 \cdot \frac{d^2}{dt^2} + (1 + \tilde{a} + \tilde{b})t \cdot \frac{d}{dt} + \tilde{a}\tilde{b}], \quad {}_2\tilde{D}_t^{-1} = a_{1,2} \cdot t(t \cdot \frac{d}{dt} + \tilde{c}),$$

$$ {}_1\tilde{D}_t^{-1} = a_{0,1} \cdot t. \tag{33}$$

When $a_{0,1} = 0$, we put $\tilde{b} = 0$ and $\tilde{c} = 0$ in the first two equations of (33). We do not consider the case when $a_{0,1} = 0$ for the third equation in it, since we consider only the differential equation satisfying Conditions 1.2 and 2.1.

We note that the point $t = 0$ is a regular point of Equation (30), and it is an irregular point of Equation (31); see Section 10.3 in Whittaker and Watson [5].

Equation (30-3) for $a_{0,0} = 0$ is the hypergeometric differential equation, whose solutions are the hypergeometric functions. Equation (30-2) for $a_{0,0} = 0$ is Kummer's differential equation, whose solutions are the confluent hypergeometric functions. Laguerre's differential equation is a special one of Kummer's differential equation; see Chapter VI in Magnus and Oberhettinger [6], and Chapter 13 in Abramowitz and Stegun [7].

We write the relations corresponding to (11) for ${}_n\tilde{D}_t^l$ as follows:

$$ {}_n\tilde{D}_t^l t^\alpha = A_{n,l}(\alpha)t^{\alpha-l}. \tag{34}$$

Then with the aid of (13), we obtain

$$ A_{2,0}(\alpha) = a_{2,2}(\alpha + a)(\alpha + b), \quad A_{1,0}(\alpha) = a_{1,1}(\alpha + c), \tag{35}$$

$$ A_{3,-1}(\alpha) = a_{2,3}(\alpha + \tilde{a})(\alpha + \tilde{b}), \quad A_{2,-1}(\alpha) = a_{1,2}(\alpha + \tilde{c}), \quad A_{1,-1}(\alpha) = a_{0,1}. \tag{36}$$

We give the solutions of the differential equations given in (30)∼(31) by applying Theorem 2.1 and Corollary 2.1. The parameters which are used in the solution, are listed in Table 1. They are obtained by comparing (35) and (36) with (24).

Theorem 2.2. We have the following solutions of Equations (30)∼(31).

(i). If $a_{0,0} \neq 0$, $a_{0,1} \neq 0$ and $a - b \notin \mathbb{Z}$, we have the pairs of solutions of (30-3), (30-2) and (30-1), respectively, given by

$$ \phi_\alpha(t) = t^\alpha \cdot {}_2F_1(\tilde{a} + \alpha, \tilde{b} + \alpha; 1 + a + b + 2\alpha; -\frac{a_{2,3}}{a_{2,2}}t), \quad \alpha = -a, -b, \tag{37}$$

$$ \phi_\alpha(t) = t^\alpha \cdot {}_1F_1(\tilde{c} + \alpha; 1 + a + b + 2\alpha; -\frac{a_{1,2}}{a_{2,2}}t), \quad \alpha = -a, -b, \tag{38}$$

$$ \phi_\alpha(t) = t^\alpha \cdot {}_0F_1(; 1 + a + b + 2\alpha; -\frac{a_{0,1}}{a_{2,2}}t), \quad \alpha = -a, -b. \tag{39}$$

Table 1. The parameters, which determine the solutions in the form of (28)**, for Equations** (30)$\sim$(31)**.** $\times$ **indicates that no equation is considered for** $a_{0,1} = 0$ **for (30-1).**

			$a_{0,0} = 0$		$a_{0,0} \neq 0$				$a_{0,1} = 0$		$a_{0,1} \neq 0$	
differential equation	μ	q	α_1	α_2	α_1	α_2	ν	p	β_1	β_2	β_1	β_2
(30-3)	$a_{2,2}$	1	0	$-a$	$-a$	$-b$	$a_{2,3}$	2	0	$-\tilde{a}$	$-\tilde{a}$	$-\tilde{b}$
(30-2)	$a_{2,2}$	1	0	$-a$	$-a$	$-b$	$a_{1,2}$	1	0		$-\tilde{c}$	
(30-1)	$a_{2,2}$	1	0	$-a$	$-a$	$-b$	$a_{0,1}$	0	$\times$	$\times$		
(31)	$a_{1,1}$	0	0		$-c$		$a_{2,3}$	2	0	$-\tilde{a}$	$-\tilde{a}$	$-\tilde{b}$

(ii). If $a_{0,0} \neq 0$ and $a_{0,1} \neq 0$, we have only one solution of (31) given by

$$\phi_{-c}(t) = t^{-c} \cdot {}_2F_0(\tilde{a} - c, \tilde{b} - c; ; -\frac{a_{2,3}}{a_{1,1}} t). \tag{40}$$

This function is a polynomial when $\tilde{a} - c \in \mathbb{Z}_{<1}$ or $\tilde{b} - c \in \mathbb{Z}_{<1}$. If such is not the case, the solution is an infinite series which has zero radius of convergence.

(iii). If $a_{0,0} = 0$, $a_{0,1} \neq 0$ and $-a \notin \mathbb{Z}$, we have the pairs of solutions of (30-3), (30-2) and (30-1), respectively, given by

$$\phi_0(t) = {}_2F_1(\tilde{a}, \tilde{b}; 1 + a; -\frac{a_{2,3}}{a_{2,2}} t), \quad \phi_{-a}(t) = t^{-a} \cdot {}_2F_1(\tilde{a} - a, \tilde{b} - a; 1 - a; -\frac{a_{2,3}}{a_{2,2}} t); \tag{41}$$

$$\phi_0(t) = {}_1F_1(\tilde{c}; 1 + a; -\frac{a_{1,2}}{a_{2,2}} t), \quad \phi_{-a}(t) = t^{-a} \cdot {}_1F_1(\tilde{c} - a; 1 - a; -\frac{a_{1,2}}{a_{2,2}} t); \tag{42}$$

$$\phi_0(t) = {}_0F_1(; 1 + a; -\frac{a_{1,1}}{a_{2,2}} t), \quad \phi_{-a}(t) = t^{-a} \cdot {}_0F_1(; 1 - a; -\frac{a_{0,1}}{a_{2,2}} t), \tag{43}$$

which are (37)$\sim$(39) for $b = 0$.

(iv). When $a_{0,0} = 0$ and $a_{0,1} \neq 0$, we have only one solution of (31) given by

$$\phi_0(t) = {}_2F_0(\tilde{a}, \tilde{b}; ; -\frac{a_{2,3}}{a_{1,1}} t), \tag{44}$$

which is (40) for $c = 0$.

(v). If $a_{0,0} \neq 0$, $a_{0,1} = 0$ and $a - b \notin \mathbb{Z}$, we have the pairs of solutions of (30-3) and (30-2), respectively, given by (37) for $\tilde{b} = 0$, and by (38) for $\tilde{c} = 0$.

(vi). If $a_{0,0} \neq 0$ and $a_{0,1} = 0$, we have only one solution of (31) given by (40) for $\tilde{b} = 0$.

For Equation (30), α_1 and α_2 are given in Table 1. Remarks are given on the cases of $\alpha_1 - \alpha_2 \in \mathbb{Z}$, in Section 4.

2.3 Remarks on the solutions of equations (30)$\sim$(31)

Remark 2.2. Equation (31) is equal to Equation (30-3) for $a_{2,2} = 0$. Accepting that b in (37) and a in (41) denote $\frac{a_{1,1}}{a_{2,2}}$, we note that (i) the solution $\phi_{-a}(t)$ given in (37) and (41), of (30-3), tend to $\phi_{-\tilde{a}}(t)$ in (40), and to $\phi_0(t)$ in (44), respectively, as $a_{2,2} \to 0$, and (ii) the second factor on the righthand side of the equation for $\phi_{-b}(t)$ in (37) and the righthand side of the equation for $\phi_0(t)$ in (41) diverge, as $a_{2,2} \to 0$.

Remark 2.3. Equation (30-2) is equal to Equation (30-3) for $a_{2,3} = 0$ and $a_{1,2} \neq 0$. Accepting that $\tilde{b}$ denotes $\frac{a_{1,2}}{a_{2,3}}$, we note that $\tilde{a}$ approaches $\tilde{c}$, and the solutions given in (37) and (41) of (30-3) tend to the corresponding ones in (38) and (42), respectively, as $a_{2,3} \to 0$.

Remark 2.4. Equation (30-1) is equal to Equation (30-2) for $a_{1,2} = 0$. Considering that $\tilde{c}$ denotes $\frac{a_{0,1}}{a_{1,2}}$, we note that the solutions given in (38) and (42) of (30-2) tend to the corresponding ones in (39) and (43), respectively, as $a_{1,2} \to 0$.

3 Reduction of Equation (21) for $l_x = 2$ and $m > 1$ to Differential Equation of the Form of (18)

In Section 2.2, the differential equations belonging to Equation (18) for $l = 0$, which is Equation (17) for $l = 0$ and $m_n = m_x = 1$, are classified as in Equations (30)∼(31), and the solutions in the form of (4) are given for these equations. In the present section, we study the differential equations belonging to Equation (21) for $l = 0$ and $m \in \mathbb{Z}_{>1}$, which are classified as in

$$ {}_2\tilde{D}_t^0 u(t) + t^{m-1} \cdot {}_n\tilde{D}_t^{-1} u(t) = 0, \quad n = 3, 2, 1, \tag{45}$$
$$ {}_1\tilde{D}_t^0 u(t) + t^{m-1} \cdot {}_3\tilde{D}_t^{-1} u(t) = 0. \tag{46}$$

Examples of (45) for $m = 2$ are Legendre's, Chebyshev's, Hermite's and Bessel's differential equations for $n = 3$, $n = 3$, $n = 2$ and $n = 1$, respectively; see Chapters IV and V in Magnus and Oberhettinger [6], p. 80 in Magnus and Oberhettinger [6] and Watson [8] respectively.

An example of (45) for $m = 3$ and $n = 1$ is Airy's differential equation; see Section 10.4 in Abramowitz and Stegun [7].

We show that the following lemmas hold valid.

Lemma 3.1. *Equations (45)∼(46) for $m = \mathbb{Z}_{>1}$ are reduced to the corresponding equations in*

$$ {}_2\tilde{D}_x^0 y(x) + {}_n\tilde{D}_x^{-1} y(x) = 0, \quad n = 3, 2, 1, \tag{47}$$
$$ {}_1\tilde{D}_x^0 y(x) + {}_3\tilde{D}_x^{-1} y(x) = 0, \tag{48}$$

by the change of variable from t to $x = t^m$, when we put $u(t) = y(x)$.

Lemma 3.2. *Let ${}_2\tilde{D}_t^0$ and ${}_3\tilde{D}_t^{-1}$ in (45) and (46) be expressed as D_t^0 and D_t^{-1}, respectively, in (10), and $\tilde{a}_{2,2}$, $\tilde{a}_{1,1}$, $\tilde{a}_{2,3}$ and $\tilde{a}_{1,2}$ denote*

$$\begin{aligned} \tilde{a}_{2,2} &= m^2 \cdot a_{2,2}, \quad \tilde{a}_{1,1} = m(m-1) \cdot a_{2,2} + m \cdot a_{1,1}, \\ \tilde{a}_{2,3} &= m^2 \cdot a_{2,3}, \quad \tilde{a}_{1,2} = m(m-1) \cdot a_{2,3} + m \cdot a_{1,2}. \end{aligned} \tag{49}$$

Then ${}_2\tilde{D}_x^0$ in (47) is expressed by D_t^0 in (10) with $a_{2,2}$, $a_{1,1}$ and t replaced by $\tilde{a}_{2,2}$, $\tilde{a}_{1,1}$ and x, respectively, and ${}_1\tilde{D}_x^0$ in (48) is given by this ${}_2\tilde{D}_x^0$ for $a_{2,2} = 0$. ${}_3\tilde{D}_x^{-1}$ in (47) and (48) is expressed as D_t^{-1} in (10) with $a_{2,3}$, $a_{1,2}$ and t replaced by $\tilde{a}_{2,3}$, $\tilde{a}_{1,2}$ and x, respectively, and ${}_2\tilde{D}_x^{-1}$ and ${}_1\tilde{D}_x^{-1}$ are given by this ${}_3\tilde{D}_x^{-1}$ for $a_{2,3} = 0$, and for $a_{2,3} = 0$ and $a_{1,2} = 0$, respectively. In these replacements, $a_{0,0}$ and $a_{1,1}$ are not changed.

Proof. We change the variable t to $x = t^m$ for $m \in \mathbb{Z}_{>1}$, and put $u(t) = y(x)$. Then

$$\frac{d}{dt}u(t) = mt^{m-1} \cdot \frac{d}{dx}y(x), \quad \frac{d^2}{dt^2}u(t) = m(m-1)t^{m-2} \cdot \frac{d}{dx}y(x) + m^2 t^{2m-2} \cdot \frac{d^2}{dx^2}y(x). \tag{50}$$

Theorem 3.1. Let $\tilde{a}_{2,2}$, $\tilde{a}_{1,1}$, $\tilde{a}_{2,3}$ and $\tilde{a}_{1,2}$ be given as in (49). The solutions of (47) and (48) are obtained from the corresponding solutions $\phi_\alpha(t)$ given in Theorem 2.2 of (30) and (31), by replacing $a_{2,2}$, $a_{1,1}$, $a_{2,3}$, $a_{1,2}$ and t, by $\tilde{a}_{2,2}$, $\tilde{a}_{1,1}$, $\tilde{a}_{2,3}$, $\tilde{a}_{1,2}$ and x, respectively. We then obtain the solutions of (45) and (46) by putting $u(t) = \phi_\alpha(t^m)$.

Lemmas 3.1 and 3.2 are applied to Hermite's and Bessel's differential equation, respectively, in Morita and Sato [4].

3.1 Parabolic cylinder function

The parabolic cylinder functions are the solutions of the following differential equation:

$$\frac{d^2}{dt^2}u(t) \mp \frac{1}{4}t^2 u(t) - au(t) = 0, \tag{51}$$

see Chapter VI, Section 4 in [6] and Chapter 19 in [7]. This takes the form:

$$({}_0\tilde{D}_t^2 + {}_0\tilde{D}_t^0 + {}_2\tilde{D}_t^{-2})u(t) = 0. \tag{52}$$

If we put $t^2 = x$ and $u(t) = y(x)$, then by using (50), we obtain the equation for $y(x)$ as follows:

$$4xy'' + 2y' \mp \frac{1}{4}xy - ay = 0, \tag{53}$$

which takes the form

$$({}_1\tilde{D}_x^1 + {}_0\tilde{D}_x^0 + {}_1\tilde{D}_x^{-1})y(x) = 0. \tag{54}$$

When the upper sign is adopted in (51), we introduce $z(x)$ by $y(x) = e^{\pm x}z(x)$. We then see that the function $z(x)$ satisfies

$$4xz'' + 2z' \mp 2xz' \mp \frac{1}{2}z - az = 0, \tag{55}$$

which takes the form of (47).

4 Analytic Continuation of Solution

We now consider the problem of solving Bessel's differential equation, whose complementary solutions are the Bessel function $J_n(t)$ and the Bessel function of the second kind $Y_n(t)$ for $n \in \mathbb{Z}_{>-1}$. $J_n(t)$ takes the form (4), and $Y_n(t)$ does not take the form (4). The argument adopted in obtaining $Y_n(t)$ in Section 3.581 of Watson [8], is as follows.

We assume that we do not know the solution $\psi_0(t)$ of a differential equation with coefficients $\{c_j\}$, but we know the solution $\psi_\delta(t)$ for the differential equation with coefficients $\{c_j + \delta d_j\}$ for $\delta \in \mathbb{C}$. If the solution is an analytic function of δ, and if the limit $\psi_0(t) = \lim_{\delta\to 0}\psi_\delta(t)$ exists, it is a solution. In the present section, we consider Equation (18). We assume the following condition.

Condition 4.1. One of the following three conditions: (i) $l = 2$, (ii) $l = 1$ and $a_{2,1} \neq 0$, and (iii) $l = 0$ and $a_{2,2} \neq 0$, is satisfied.

This is the condition that $A_l(\alpha)$ given by (13) is a quadratic function of α.

Definition 4.1. Let $A_l(\alpha)$ be a quadratic function of α. If $A_l(\alpha) = 0$ has two distinct roots, we call them α_1^* and α_2^*, so that Re $\alpha_1^* \geq$ Re α_2^*. If it has only one root, we denote it α_1^* as well as α_2^*.

Proposition 4.1. *Let Equation* (18) *satisfy Condition 4.1, and Definition 4.1 be adopted. Then we have two solutions of the equation, one of which is $\phi_{\alpha_1^*}(t)$. The second one is given as follows.*

(i). *If $A_l(\alpha) = 0$ has two distinct roots, and $\alpha_1^* - \alpha_2^* \notin \mathbb{Z}_{>-1}$, then the second one is $\phi_{\alpha_2^*}(t)$.*

(ii). *If $A_l(\alpha) = 0$ has two distinct roots, and $\alpha_1^* - \alpha_2^* = n \in \mathbb{Z}_{>-1}$, then the second one is $\phi_{\alpha_2^*}(t)$, or takes of the form*

$$\phi_{\alpha_1^*}(t)\log t + \phi^*_{\alpha_2^*}(t), \tag{56}$$

according to as $\phi_{\alpha_2^*}(t)$ is expressed as $t^{\alpha_2^*}$ multiplied by a polynomial of degree less than n, or not.

(iii). *If $A_l(\alpha) = 0$ has only one root, then the second solution is of the form* (56).

$\phi^*_{\alpha^*_2}(t)$ *in* (56) *takes the form* (4) *with* $\alpha = \alpha^*_2$.

This proposition is a consequence of Theorem 2.1, excluding the cases when the solution in the form (56) appears.

Remark 4.1. Solutions of the form (56) are found for the confluent hypergeometric function and the hypergeometric function, in Sections 13.1.6 and 15.5.16∼21, respectively, of Abramowitz and Stegun [7].

5 Solution of Equation (18) for $l_x = 2$ by Means of the AC-Laplace Transform

5.1 Preliminary formulas of the AC-Laplace transform

Definition 5.1. Let $f(a, t)$ be such a function of $t \in \mathbb{R}_{>0}$ and $a \in D_0 \subset \mathbb{C}$, that

(i). $f(a, t)$ is analytic as a function of a in the domain D_0 for fixed $t \in \mathbb{R}_{>0}$,

(ii). the Laplace transform $\hat{f}(a, s)$ defined by

$$\hat{f}(a, s) := \mathcal{L}[f(a, t)] = \int_0^\infty f(a, t)e^{-st}dt, \tag{57}$$

exists if $a \in D_1 \subset D_0$ and is analytic as a function of a in the domain D_1,

(iii). $\hat{f}(a, s)$ defined by (57) is analytic as a function of a in the domain D_0.

Then we call the analytic continuation as a function of a of $\hat{f}(a, s)$ to the domain D_0, the AC-Laplace transform of $f(a, t)$ and denote it as $\hat{f}(a, s) = \mathcal{L}_H[f(a, t]$ for $a \in D_0$.

In Section 5.2, we study Equation (30) by using the AC-Laplace transform of $u(t)$ expressed by (3). We first note that the Laplace transform of $g_\nu(t) = \frac{t^{\nu-1}}{\Gamma(\nu)}$ is given by $\mathcal{L}[g_\nu(t)] = s^{-\nu}$ for $\nu \in \mathbb{C}$ satisfying Re $\nu > 0$. We call the analytic continuation of this $\mathcal{L}[g_\nu(t)]$ as a function of ν to $\nu \in \mathbb{C}\backslash\mathbb{Z}_{<1}$ the AC-Laplace transform, that is given by $\hat{g}_\nu(s) := \mathcal{L}_H[g_\nu(t)] = s^{-\nu}$ for $\nu \in \mathbb{C}\backslash\mathbb{Z}_{<1}$.

As in [3, 4], we assume that $u(t)$ satisfies Condition 1.1 and is expressed as (3). Then its AC-Laplace transform $\hat{u}(s)$ is expressed as

$$\hat{u}(s) = \sum_{\nu \in S} u_{\nu-1}s^{-\nu}. \tag{58}$$

The derivative of $g_\nu(t)$ of order $l \in \mathbb{Z}_{>0}$ is calculated by

$$\frac{d^l}{dt^l}g_\nu(t) = \begin{cases} g_{\nu-l}(t), & \nu - l \in \mathbb{C}\backslash\mathbb{Z}_{<1}, \\ 0, & \nu - l \in \mathbb{Z}_{<1}. \end{cases} \tag{59}$$

The AC-Laplace transform of $\frac{d^l}{dt^l}g_\nu(t)$ is given by

$$\mathcal{L}_H[\frac{d^l}{dt^l}g_\nu(t)] = s^{l-\nu} - \langle s^{l-\nu}\rangle_0, \tag{60}$$

where

$$\langle s^{l-\nu}\rangle_0 = \begin{cases} s^k, & k = l - \nu \in \mathbb{Z}_{>-1}, \\ 0, & l - \nu \notin \mathbb{Z}_{>-1}. \end{cases} \tag{61}$$

We note here the formulas:

$$t\cdot g_\nu(t)=t\cdot\frac{t^{\nu-1}}{\Gamma(\nu)}=\nu\cdot\frac{t^\nu}{\Gamma(\nu+1)}=\nu g_{\nu+1}(t), \tag{62}$$

$$-\frac{d}{ds}s^{-\nu}=\nu s^{-\nu-1}=\nu\mathcal{L}_H[g_{\nu+1}(t)]. \tag{63}$$

By using these, we confirm that

$$\mathcal{L}_H[t^m g_\nu(t)]=(-1)^m\frac{d^m}{ds^m}s^{-\nu}. \tag{64}$$

By applying these formulas, we obtain

Lemma 5.1. *Let $m\in\mathbb{Z}_{>0}$, $l\in\mathbb{Z}_{>0}$, $u(t)$ be expressed by* (3) *and $\hat{u}(s):=\mathcal{L}_H[u(t)]$. Then*

$$\mathcal{L}_H[t^m u(t)]=(-1)^m\frac{d^m}{ds^m}\hat{u}(s), \tag{65}$$

$$\mathcal{L}_H[\frac{d^l}{dt^l}u(t)]=s^l\hat{u}(s)-\langle s^l\hat{u}(s)\rangle_0, \tag{66}$$

$$\mathcal{L}_H[t^m\frac{d^l}{dt^l}u(t)]=(-1)^m\frac{d^m}{ds^m}[s^l\hat{u}(s)]-(-1)^m\frac{d^m}{ds^m}\langle s^l\hat{u}(s)\rangle_0, \tag{67}$$

where

$$\langle s^l\hat{u}(s)\rangle_0=\sum_{k=0}^{l-1}u_{l-k-1}s^k. \tag{68}$$

In particular,

$$\langle s\hat{u}(s)\rangle_0=u_0,\quad\langle s^2\hat{u}(s)\rangle_0=u_0s+u_1,\quad\langle s^3\hat{u}(s)\rangle_0=u_0s^2+u_1s+u_2. \tag{69}$$

5.2 Solutions of equation (30) for $a_{0,0}=0$

Equation (30-2) for $a_{0,0}=0$ is given by Equation (5) in Morita and Sato [4], with c, a and b replaced by $1+a$, $\tilde{c}$ and $-\frac{a_{1,2}}{a_{2,2}}$, respectively. In [4], the AC-Laplace transform of Equation (5) in [4] is given by (50) in [4]. The corresponding AC-Laplace transform of (30-2) for $a_{0,0}=0$, is given by

$$-\frac{d}{ds}[(s^2+\frac{a_{1,2}}{a_{2,2}}s)\hat{u}(s)]+[(1+a)s+\tilde{c}\frac{a_{1,2}}{a_{2,2}}]\hat{u}(s)=-au_0. \tag{70}$$

The differential equation is of the first order, and its complementary solution is $\hat{\phi}_{-a}(s)$ which is the AC-Laplace transform of $\phi_{-a}(t)$ given in (42). By applying the inverse AC-Laplace transform to the obtained $\hat{\phi}_{-a}(s)$, we obtain the solution $\phi_{-a}(t)$ of (30-2). In Morita and Sato [3] the other solution is given by obtaining a particular solution of the differential equation for $\hat{u}(s)$, $\hat{\phi}_0(s)$, which is the Laplace transform of $\phi_0(t)$ given in (42). In Morita and Sato [4] after obtaining $\phi_{-a}(t)$, $\phi_0(t)$ is obtained by using the forthcoming Lemma 5.4 for (30-2).

Equation (30-3) for $a_{0,0}=0$ is given by Equation (10) in Morita and Sato [4], if c, a and b, t and $\frac{d}{dt}$ are replaced by $1+a$, $\tilde{a}$, $\tilde{b}$, $\beta_3 t$ and $\frac{1}{\beta_3}\frac{d}{dt}$, respectively, where $\beta_3=-\frac{a_{2,3}}{a_{2,2}}$. Corresponding to Equation (60) in [4], the AC-Laplace transform of (30-3) for $a_{0,0}=0$ is given by

$$\beta_3\frac{d^2}{ds^2}[s^2\hat{u}(s)]-\frac{d}{ds}[(s^2+\beta_3(\tilde{a}+\tilde{b}+1)s)\hat{u}(s)]+[(1+a)s+\beta_3\tilde{a}\tilde{b}]\hat{u}(s)=-au_0. \tag{71}$$

Corresponding to Equations (11) and (12), we have (41).

Remark 5.1. In obtaining the complementary solution of (71) by the method given in Morita and Sato [4], we use Equation (64) in [4], where $\lambda(\lambda-1)$ should be replaced by $\lambda(\lambda+1)$.

Equation (30-1) for $a_{0,0}=0$ is given by putting $a_{1,2}=0$ and $a_{1,2}\cdot\tilde{c}=a_{0,1}$ in (30-2) for $a_{0,0}=0$, and hence the AC-Laplace transform of (30-1) for $a_{0,0}=0$ is obtained from (70) as

$$-\frac{d}{ds}[s^2\hat{u}(s)]+[(1+a)s+\frac{a_{0,1}}{a_{2,2}}]\hat{u}(s)=-au_0. \tag{72}$$

The complementary solution of this equation is

$$\hat{u}(s)=Cs^{1+a}e^{\beta_1 s^{-1}}=Cs^{1+a}\sum_{k=0}^{\infty}\frac{\beta_1^k}{k!}s^{-k}, \tag{73}$$

where $\beta_1=-\frac{a_{0,1}}{a_{2,1}}$. By the inverse AC-Laplace transform, we obtain $\phi_{-a}(t)$ given in (43), by choosing $C=\Gamma(1-a)$. After obtaining $\phi_{-a}(t)$, $\phi_0(t)$ is obtained by using the forthcoming Lemma 5.4, for (30-1).

5.3 Solutions of equations (30)∼(31) for $a_{0,0}\neq 0$

In Morita and Sato [3, 4], we obtain the solutions (41) and (42) of the differential equation (30-3) and (30-2) for $a_{0,0}=0$ by the method of AC-Laplace transform. The solution (43) of the differential equation (30-1) for $a_{0,0}=0$ is shown to be obtained by that method at the end of last section. We now present a method by which the solutions (37)∼ (39) for $a_{0,0}\neq 0$ are obtained with the aid of solutions (41)∼ (43) for $a_{0,0}=0$.

In Section 2.2, we gave the solutions of Equations (30)∼(31). We now study related equations, which are

$$_2\hat{D}_t^0 y(t)+{}_n\hat{D}_t^{-1}y(t)=0,\quad n=3,2,1, \tag{74}$$

$$_1\hat{D}_t^0 y(t)+{}_3\hat{D}_t^{-1}y(t)=0, \tag{75}$$

where

$$_2\hat{D}_t^0=a_{2,2}[t^2\cdot\frac{d^2}{dt^2}+(1+2\alpha+a+b)t\cdot\frac{d}{dt}+(\alpha+a)(\alpha+b)],\quad {}_1\hat{D}_t^0=a_{1,1}(t\cdot\frac{d}{dt}+\alpha+c), \tag{76}$$

$$_3\hat{D}_t^{-1}=a_{2,3}t[t^2\cdot\frac{d^2}{dt^2}+(1+2\alpha+\tilde{a}+\tilde{b})t\cdot\frac{d}{dt}+(\alpha+\tilde{a})(\alpha+\tilde{b})],\quad {}_2\hat{D}_t^{-1}=a_{1,2}t(t\cdot\frac{d}{dt}+\alpha+\tilde{c}),$$
$$_1\hat{D}_t^{-1}=a_{0,1}t. \tag{77}$$

We call Equation (74) for $n=3,2$ and 1 as (74-3), (74-2) and (74-1), respectively.

$_n\hat{D}_t^0$ and $_n\hat{D}_t^{-1}$ given in (76) and (77) are so constructed from $_n\tilde{D}_t^0$ and $_n\tilde{D}_t^{-1}$ given in (32) and (33) that the following equations hold:

$$_n\tilde{D}_t^0[t^\alpha y(t)]=t^\alpha\cdot{}_n\hat{D}_t^0 y(t),\quad {}_n\tilde{D}_t^{-1}[t^\alpha y(t)]=t^\alpha\cdot{}_n\hat{D}_t^{-1}y(t). \tag{78}$$

Let $u(t)$ be the solution of (30-3) for $a_{0,0}\neq 0$, and $u(t)=t^{-a}y(t)$. Then $y(t)$ satisfies (74-3) for $\alpha=-a$, and then this equation is Equation (30-3) for $b=0$, with $\tilde{a}$, $\tilde{b}$, a and $u(t)$ replaced by $\tilde{a}-a$, $\tilde{b}-a$, $b-a$ and $y(t)$, respectively. Then $y(t)$ is given by (41) with the same replacement. Thus we obtain the solution of (30-3) for $a_{0,0}\neq 0$ by using the solution (41) for $a_{0,0}=0$.

The following two lemmas are consequences of this type of arguments.

Lemma 5.2. *Let* $u_3(\tilde{a}, \tilde{b}, a; -\frac{a_{2,3}}{a_{2,2}}t)$, $u_2(\tilde{c}, a; -\frac{a_{1,2}}{a_{2,2}}t)$ *and* $u_1(a; -\frac{a_{0,1}}{a_{2,2}}t)$ *be solutions of* (30-3), (30-2) *and* (30-1)*, respectively, for the case of* $a_{0,0} = 0$ *and* $a_{0,1} \neq 0$*. Then if* $\alpha = -a$ *or* $\alpha = -b$, $t^{\alpha}u_3(\alpha + \tilde{a}, \alpha + \tilde{b}, 2\alpha + a + b; -\frac{a_{2,3}}{a_{2,2}}t)$, $t^{\alpha}u_2(\tilde{c} + \alpha, 2\alpha + a + b; -\frac{a_{1,2}}{a_{2,2}}t)$ *and* $t^{\alpha}u_1(2\alpha + a + b; -\frac{a_{0,1}}{a_{2,2}}t)$, *respectively, are solutions of* (30-3), (30-2) *and* (30-1) *for* $a_{0,0} \neq 0$ *and* $a_{0,1} \neq 0$.

Lemma 5.3. *Let the solution of* (31) *for* $a_{0,0} = 0$ *and* $a_{0,1} \neq 0$ *be given by* $u_0(\tilde{a}, \tilde{b}, -\frac{a_{2,3}}{a_{0,1}}t)$*. Then the solution of* (31) *for* $a_{0,0} \neq 0$ *and* $a_{0,1} \neq 0$ *is given by* $t^{-c}u_0(\tilde{a} - c, \tilde{b} - c, -\frac{a_{2,3}}{a_{0,1}}t)$.

Remark 5.2. After (37) is obtained, (40) is obtained from it by Remark 2.2, and (44) is obtained by putting $\tilde{a} = 0$ in (40), or from (41) by Remark 2.2. When (44) is known before (40), the latter is obtained from the former with the aid of Lemma 5.3.

By putting $b = 0$ and $\alpha = -a$ in Lemma 5.2, we obtain the following lemma.

Lemma 5.4. *Let the condition in Lemma 5.2 be satisfied. Then* $t^{-a}u_3(\tilde{a} - a, \tilde{b} - a, -a; -\frac{a_{2,3}}{a_{2,2}}t)$, $t^{-a}u_2(\tilde{a} - a, -a; -\frac{a_{1,2}}{a_{2,2}}t)$ *and* $t^{-a}u_1(-a; -\frac{a_{0,1}}{a_{2,2}}t)$ *are also solutions of the respective differential equation for* $a_{0,0} = 0$ *and* $a_{0,1} \neq 0$.

In Morita and Sato [4], after obtaining $\phi_{-a}(t)$ given in (42), $\phi_0(t)$ in (42) is obtained by using this lemma for Equation (30-2) for $a_{0,0} = 0$.

6 Conclusion

In the present paper, we express the linear differential equation with polynomial coefficients in terms of blocks of classified terms, which are defined by (7), and by using (10) for the equations of the second order. The equation with only one block is Euler's differential equation, whose solution is given in Theorem 1.1. Equation (18) expresses the equations with two adjacent blocks. Except for special values of the coefficients, the solutions of these equations are obtained in the form of the generalized hypergeometric series as stated in Theorem 2.1. For Equation (18) of the second order, detailed study is presented in Section 2.2. In Section 3, for the differential equation of the second order, with two blocks which are not adjacent with each other, the solution is shown to be obtained from the solution of an equation with two adjacent blocks by a change of variable. In Section 4, comments are given on the solutions involving $\log t$.

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Competing Interests

Authors have declared that no competing interests exist.

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