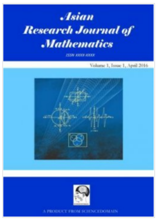

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Two Turán Type Inequalities Related to the Q-exponential Functions

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Authors' contributions

This work was carried out in collaboration between all authors. Author XKD designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Author LY managed the analyses of the study. Author CFL managed the literature searches. All authors read and approved the final manuscript.

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Abstract

In this paper, we establish some new Turán type inequalities related to the remainder of q-analogue of exponential functions. Our results are shown to be a generalization which were obtained by K. Mehrez in 2015.

Keywords: Q-exponential functions; Turán type inequalities; monotonicity.

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1 Introduction

The difference $I_n(x) = e^x - \sum_{k=0}^{n} \frac{x^k}{k!}$ for real x and positive integers n have been studied by many mathematicians. In 1943, P. K. Menon [1] proved the intriguing inequality

$$I_{n-1}(x)I_{n+1}(x) > \frac{1}{2}(I_n(x))^2, \tag{1.1}$$

which is valid for all positive integers n and for all $x > 0$. Later, H. Alzer [2] established the sharpened inequality

$$I_{n-1}(x) \cdot I_{n+1}(x) > \frac{n+1}{n+2}(I_n(x))^2, \tag{1.2}$$

for all $n \in \mathbf{N}$ and $x > 0$, and with the best possible constant $\frac{n+1}{n+2}$.

Recently, S. M. Sitnik formulated some conjectures on monotonicity of ratios for exponential series remainders. They are equivalent to conjectures on monotonicity of a ratio of Kummer hypergeometric function, see [3] and [4]. Afterwards, K. Mehrez and S. M. Sitnik proved their conjectures in [5].

In 2015, L. Yin and W. -Y. Cui [6] showed a generalization of Alzer inequality related to exponential function, and generalized it for the remainder of Maclaurin series. The main purpose of this note is to find the greatest value $C_{n,p}$, such that

$$I_{n-p}(q,z)I_{n+p}(q,z) > C_{n,p}(I_n(q,z))^2 \tag{1.3}$$

is valid for every positive z and $n, p \in \mathbf{N}$. K. Mehrez [7] deduced some sharp Turán type inequalities for the remainder of q-exponential functions in 2015. Actually, the Turán type inequalities have a more extensive literature and recently the results have been applied in problems arising from many fields such as information theory, economic theory and biophysics. For more about this subject the readers refer to [8, 9, 10, 11, 12, 13, 14, 15, 1, 3, 16, 17] and the references therein.

2 Basic Symbols and Lemmas

In this note, we fix $q \in (0, 1)$. For the definitions, notations and properties of the q-shifted factorial and the q-analogue of exponential functions, the readers may refer to [7]. Let $a \in \mathbf{R}$, the q-shifted factorial is defined by

$$(a; q)_0 = 1, (a; q)_n = \prod_{k=0}^{n-1}(1 - aq^k), (a; q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k).$$

To simplify the writing, the following compact notation is used frequency

$$(a_1, a_2, \cdots, a_p; q) = (a_1; q)_n(a_2; q)_n \cdots (a_p; q)_n, n = 0, 1, 2, \cdots.$$

Then note that for $q \to 1$ the expression $\frac{(q^a;q)_n}{(1-q)^n}$ tends to $(a)_n = a(a + 1) \cdots (a + n - 1)$.

For $q \in (0, 1)$, the q-analogue of exponential functions are given by follows

$$E(q, z) = \sum_{n=0}^{\infty} \frac{z^n}{(q, q)_n} = \frac{1}{(z; q)_\infty}, |z| < 1.$$

and

$$\mathcal{E}(q; z) = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{z^n}{(q, q)_n} = \frac{1}{(z; q)_\infty} = \prod_{k=0}^{\infty}(1 - zq^k), z \in C.$$

We denote by $I_n(q;z)$ and $\mathcal{I}_n(q;z)$ the differences

$$I_n(q;z) = E(q,z) - \sum_{k=0}^{n} \frac{z^k}{(q,q)_k}, 0 < z < 1,$$

and

$$\mathcal{I}_n(q;z) = \mathcal{E}(q;z) - \sum_{k=0}^{n} q^{\frac{k(k-1)}{2}} \frac{z^k}{(q,q)_k}, z > 0.$$

Lemma 2.1 ([5]). *Let $\{a_n\}$ and $\{b_n\}, (n = 0,1,2,\cdots)$ be real numbers such that $b_n > 0$ and $\{\frac{a_n}{b_n}\}_{n\geq 0}$ is increasing(decreasing), then $\{\frac{a_0+a_1+\cdots+a_n}{b_0+b_1+\cdots+b_n}\}$ is increasing(decreasing).*

Lemma 2.2 ([5]). *Let $\{a_n\}$ and $\{b_n\}, (n = 0,1,2,\cdots)$ be real numbers and let the power series $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ be convergent if $|x| < r$. If $b_n > 0, (n = 0,1,2,\cdots)$ and the sequence $\{\frac{a_n}{b_n}\}_{n\geq 0}$ is (strictly)increasing(decreasing), then the function $\frac{A(x)}{B(x)}$ is also (strictly) increasing(decreasing) on $[0,r)$.*

3 Main Results

Theorem 3.1. *For every $n, p \in \mathbf{N}$, $q \in (0,1)$ and $0 < z < 1$. The function*

$$E(n,p,q,z) = \frac{I_{n-p}(q;z) I_{n+p}(q;z)}{I_n^2(q;z)} \tag{3.1}$$

is strictly increasing on $(0,\infty)$. As a result, we have the following Turán type inequalities

$$\frac{I_{n-p}(q;z) I_{n+p}(q;z)}{I_n^2(q;z)} > \frac{(1-q^{n-p+2})\cdots(1-q^{n+1})}{(1-q^{n+2})\cdots(1-q^{n+p+1})} \tag{3.2}$$

where the constant $\frac{(1-q^{n-p+2})\cdots(1-q^{n+1})}{(1-q^{n+2})\cdots(1-q^{n+p+1})}$ is best possible.

Proof.

$$E(n,p,q,z) = \frac{I_{n-p}(q;z) I_{n+p}(q;z)}{I_n^2(q;z)} = \frac{\sum_{k=n-p+1}^{\infty} \frac{z^k}{(q,q)_k} \sum_{k=n+p+1}^{\infty} \frac{z^k}{(q,q)_k}}{\left(\sum_{k=n+1}^{\infty} \frac{z^k}{(q,q)_k}\right)^2}$$

$$= \frac{\sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{z^{2n+2+k}}{(q,q)_{n+p+1+k-j}(q,q)_{n-p+1+j}}}{\sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{z^{2n+2+k}}{(q,q)_{n+1+k-j}(q,q)_{n+1+j}}} = \frac{\sum_{k=0}^{\infty} H_{p,q,k} z^{2n+2+k}}{\sum_{k=0}^{\infty} G_{p,q,k} z^{2n+2+k}} \tag{3.3}$$

where $H_{p,q,k} = \sum_{j=0}^{k} \frac{1}{(q,q)_{n+p+1+k-j}(q,q)_{n-p+1+j}}$, $G_{p,q,k} = \sum_{j=0}^{k} \frac{1}{(q,q)_{n+1+k-j}(q,q)_{n+1+j}}$.

Define sequences $\{A_{n,p,q,j}\}$,$\{B_{n,p,q,j}\}$ and $\{C_{n,p,q,j}\}$ by

$$A_{n,p,q,j} = \frac{1}{(q,q)_{n+p+1+k-j}(q,q)_{n-p+1+j}},$$

$$B_{n,p,q,j} = \frac{1}{(q,q)_{n+1+k-j}(q,q)_{n+1+j}},$$

and

$$C_{n,p,q,j} = \frac{A_{n,p,q,j}}{B_{n,p,q,j}} = \frac{(q,q)_{n+1+k-j}(q,q)_{n+1+j}}{(q,q)_{n+p+1+k-j}(q,q)_{n-p+1+j}}.$$

Simple computation yields

$$\begin{aligned}\frac{C_{n,p,q,j+1}}{C_{n,p,q,j}} &= \frac{(q,q)_{n+1+k-j-1}(q,q)_{n+1+j+1}(q,q)_{n+p+1+k-j}(q,q)_{n-p+1+j}}{(q,q)_{n+p+1+k-j-1}(q,q)_{n-p+2+j}(q,q)_{n+1+k-j}(q,q)_{n+1+j}} \\ &= \frac{(1-q^{n+2+j})(1-q^{n+p+k-j+1})}{(1-q^{n+2-p+j})(1-q^{n+1+k-j})} > 1. \end{aligned} \tag{3.4}$$

This implies that the sequence $C_{n,p,q,j}$ is strictly increasing to j. By using Lemma 2.1 and Lemma 2.2, we easily obtain the function $E(n,p,q,z)$ is strictly increasing on $(0,\infty)$.

Finally, from limit identity

$$\lim_{z\to 0^+} \frac{I_{n-p}(q;z)I_{n+p}(q;z)}{I_n^2(q;z)} = \frac{(q,q)_{n+1}^2}{(q,q)_{n+p+1}(q,q)_{n-p+1}} = \frac{(1-q^{n-p+2})\cdots(1-q^{n+1})}{(1-q^{n+2})\cdots(1-q^{n+p+1})}, \tag{3.5}$$

we complete the proof. □

Corollary 3.1. *Taking $p = 1$, we have the following Turán type inequality*

$$\frac{I_{n-1}(q;z)I_{n+1}(q;z)}{I_n^2(q;z)} > \frac{1-q^{n+1}}{1-q^{n+2}} \tag{3.6}$$

where the constant $\frac{1-q^{n+1}}{1-q^{n+2}}$ can not be replaced by a greater number.

Remark 3.1. These results have been shown by K. Mehrez in [7]. Here an alternative proof is provided based on a different method.

Theorem 3.2. *For every $n,p \in \mathbf{N}$, $q \in (0,1)$ and $z > 0$. The function*

$$\mathcal{E}(n,p,q,z) = \frac{\mathcal{I}_{n-p}(q;z)\mathcal{I}_{n+p}(q;z)}{\mathcal{I}_n^2(q;z)}$$

is strictly increasing on $(0,\infty)$. As a result, we have the following Turán type inequalities

$$\frac{\mathcal{I}_{n-p}(q;z)\mathcal{I}_{n+p}(q;z)}{\mathcal{I}_n^2(q;z)} > \frac{(1-q^{n-p+2})\cdots(1-q^{n+1})q^{p^2}}{(1-q^{n+2})\cdots(1-q^{n+p+1})} \tag{3.7}$$

where the constant $\frac{(1-q^{n-p+2})\cdots(1-q^{n+1})q^{p^2}}{(1-q^{n+2})\cdots(1-q^{n+p+1})}$ is best possible.

Proof. Direct computation results in

$$\mathcal{E}(n,p,q,z) = \frac{\mathcal{I}_{n-p}(q;z)\mathcal{I}_{n+p}(q;z)}{\mathcal{I}_n^2(q;z)} = \frac{\sum\limits_{k=n-p+1}^{\infty} \frac{q^{\frac{k(k+1)}{2}} z^k}{(q,q)_k} \sum\limits_{k=n+p+1}^{\infty} \frac{q^{\frac{k(k+1)}{2}} z^k}{(q,q)_k}}{\left(\sum\limits_{k=n+1}^{\infty} \frac{q^{\frac{k(k+1)}{2}} z^k}{(q,q)_k}\right)^2}$$

$$= \frac{\sum\limits_{k=0}^{\infty}\sum\limits_{j=0}^{k} \frac{q^{\frac{(n+p+1+k-j)(n+p+2+k-j)}{2}} q^{\frac{(n-p+1+j)(n-p+2+j)}{2}} z^{2n+2+k}}{(q,q)_{n+p+1+k-j}(q,q)_{n-p+1+j}}}{\sum\limits_{k=0}^{\infty}\sum\limits_{j=0}^{k} \frac{q^{\frac{(n+1+k-j)(n+2+k-j)}{2}} q^{\frac{(n+1+j)(n+2+j)}{2}} z^{2n+2+k}}{(q,q)_{n+1+k-j}(q,q)_{n+1+j}}}$$

$$= \frac{\sum_{k=0}^{\infty} H_{p,q,k} z^{2n+2+k}}{\sum_{k=0}^{\infty} G_{p,q,k} z^{2n+2+k}}, \tag{3.8}$$

where

$$H_{p,q,k} = \sum_{j=0}^{k} \frac{q^{\frac{(n+p+1+k-j)(n+p+2+k-j)}{2}} q^{\frac{(n-p+1+j)(n-p+2+j)}{2}} z^{2n+2+k}}{(q,q)_{n+p+1+k-j}(q,q)_{n-p+1+j}},$$

$$G_{p,q,k} = \sum_{j=0}^{k} \frac{q^{\frac{(n+1+k-j)(n+2+k-j)}{2}} q^{\frac{(n+1+j)(n+2+j)}{2}} z^{2n+2+k}}{(q,q)_{n+1+k-j}(q,q)_{n+1+j}},$$

Define sequences $\{\mathcal{A}_{n,p,q,j}\}$, $\{\mathcal{B}_{n,p,q,j}\}$ and $\{\mathcal{C}_{n,p,q,j}\}$ by

$$\mathcal{A}_{n,p,q,j} = \frac{q^{\frac{(n+p+1+k-j)(n+p+2+k-j)}{2}} q^{\frac{(n-p+1+j)(n-p+2+j)}{2}} z^{2n+2+k}}{(q,q)_{n+p+1+k-j}(q,q)_{n-p+1+j}},$$

$$\mathcal{B}_{n,p,q,j} = \frac{q^{\frac{(n+1+k-j)(n+2+k-j)}{2}} q^{\frac{(n+1+j)(n+2+j)}{2}} z^{2n+2+k}}{(q,q)_{n+1+k-j}(q,q)_{n+1+j}},$$

and

$$\mathcal{C}_{n,p,q,j} = \frac{\mathcal{A}_{n,p,q,j}}{\mathcal{B}_{n,p,q,j}}.$$

By easy computation, we have

$$\frac{\mathcal{C}_{n,p,q,j+1}}{\mathcal{C}_{n,p,q,j}} = \frac{(1-q^{n+2+j})(1-q^{n+p+k-j+1})}{(1-q^{n+2-p+j})(1-q^{n+1+k-j})q^2} > 1. \tag{3.9}$$

So, the sequence $\mathcal{C}_{n,p,q,j}$ is strictly increasing to j. By using Lemma 2.1 and Lemma 2.2, we get that the function $\mathcal{E}(n,p,q,z)$ is strictly increasing on $(0,\infty)$.

Finally, from limit identity

$$\lim_{z\to 0^+} \frac{\mathcal{I}_{n-p}(q;z)\mathcal{I}_{n+p}(q;z)}{\mathcal{I}_n^2(q;z)} = \frac{q^{\frac{(n+p+1)(n+p+2)}{2}} q^{\frac{(n-p+1)(n-p+2)}{2}} (q,q)_{n+1}^2}{q^{(n+1)(n+2)}(q,q)_{n+p+1}(q,q)_{n-p+1}}$$

$$= \frac{(1-q^{n-p+2})\cdots(1-q^{n+1})q^{p^2}}{(1-q^{n+2})\cdots(1-q^{n+p+1})}, \tag{3.10}$$

the proof is complete. □

Corollary 3.2. *Taking* $p=1$*, we have the following Turán type inequalities*

$$\frac{\mathcal{I}_{n-1}(q;z)\mathcal{I}_{n+1}(q;z)}{\mathcal{I}_n^2(q;z)} > \frac{q-q^{n+2}}{1-q^{n+2}} \tag{3.11}$$

where the constant $\frac{q-q^{n+2}}{1-q^{n+2}}$ *can not be replaced by a greater number.*

4 Conclusion

In this paper, we mainly establish two monotonic results related to the remainder of q-analogue of exponential functions, and some new Turán type inequalities such as theorem 3.1 and 3.2 were obtained. Our results are shown to be a generalization which were obtained by K. Mehrez in 2015.

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Competing Interests

Authors have declared that no competing interests exist.

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