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Computational Method for the Simulation of Duffing Oscillators

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Authors' contributions

This work was carried out in collaboration between all authors. Authors JS and SEG designed the methodology and implemented the method derived on Duffing oscillators with the aid of softwares. Author DJZ analyzed the basic properties of the method derived. All authors read and approved the final manuscript.

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ABSTRACT

A one-step computational method is proposed for the simulation of Duffing oscillators in this research. In achieving this, power series was adopted as a basis function in the derivation of the method. The integration was carried out within a one-step interval, where the interval was partitioned at four different points. The computational method developed was applied on some Duffing equations and from the results obtained; it was evident that the method developed is computationally reliable.

Keywords: Computational method; damping; Duffing oscillator; nonlinear; simulations.

2010 AMS subject classification: 65L05, 65L06, 65D30.

1. INTRODUCTION

Duffing equation is one of the most significant and classical nonlinear ordinary differential equations in view of its diverse applications in science and engineering, [1]. Little wonder, it has received remarkable attention due to its variety of applications in science and engineering. The

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Duffing oscillators are applied in weak signal detection [2], magneto-elastic mechanical systems [3], large amplitude oscillation of centrifugal governor systems [4], nonlinear vibration of beams and plates [5], fluid flow induced vibration [6], among others. Given its characteristic of oscillation and chaotic nature, many scientists are inspired by this nonlinear differential equation since it replicates similar dynamics in our natural world.

In this paper, we shall consider a computational method for the simulation of Duffing oscillators of the form;

$$y''(t)+\eta y'(t)+\mu y(t)+\gamma y^{3}(t)=f(t) \quad (1)$$

with initial conditions,

$$y(0)=\alpha,\ y'(0)=\beta \quad (2)$$

where $\eta,\ \mu, \gamma,\ \alpha\ and\ \beta$ are real constants and $f(t)$ is a real-valued function. We shall assume that equation (1) satisfy the existence and uniqueness theorem stated below.

Theorem 1.1 [7]

Let,

$$u^{(n)}=f(x,u,u',...,u^{(n-1)}),\ u^{(k)}(x_0)=c_k \quad (3)$$

$k=0,1,...,(n-1)$ *,* u *and* f *are scalars. Let* $\Re$ *be the region defined by the inequalities* $x_0\le x\le x_0+a,\ \left|s_j-c_j\right|\le b,\ j=0,1,...,(n-1)$ *,* $(a>0,b>0)$ *. Suppose the function* $f(x,s_0,s_1,...,s_{n-1})$ *is defined in* $\Re$ *and in addition:*

(i) f *is non-negative and non-decreasing in each of* $x,s_0,s_1,...,s_{n-1}$ *in* $\Re$

(ii) $f(x,c_0,c_1,...,c_{n-1})>0$ *, for* $x_0\le x\le x_0+a$ *, and*

(iii) $c_k\ge 0,\ k=0,1,...,n-1$

Then, the initial value problem (1) and (2) has a unique solution in $\Re$*.*

Several methods have been proposed in literature for simulating problems of the form (1). These methods include; Hybrid method [1], Laplace decomposition method [8], restarted Adomian decomposition method [9], differential transform method [10], modified differential transform method [11], improved Taylor matrix method [12], variational iteration method [13,14], modified variational iteration method [15], Trigonometrically fitted Obrechkoff method [16], among others. The most recent of these works is the development of hybrid block method for the simulation of problems of the form (1), see [1] for details.

It is important to note that the Duffing equation is a simple model that shows different types of oscillations such as chaos and limit cycles. The terms associated with the system in equation (1) as given by [1] are;

$y'(t)$: Small damping

η: Ratio (coefficient) of viscous damping (it controls the size of damping)

$\mu y(t)+\gamma y^{3}(t)$: Nonlinear restoring force acting like a hard spring (with μ controlling the size of stiffness and γ controlling the size of nonlinearity)

$f(t)$: Small periodic force

Duffing equations are routinely associated with damping in physical systems [1], where damping is defined as an influence within or upon oscillatory system that has the effect of reducing, restricting or preventing its oscillation.

2. MATHEMATICAL FORMULATION OF THE COMPUTATIONAL METHOD

We shall formulate a discrete computational method (which is an extension of the earlier work of [1]) for the simulation of equation (1). The author in [1] partitioned the one-step interval at three different points. However, in this research, the one-step interval shall be partitioned at four different points. This will enable us to develop a more accurate method that will be used for the simulation of equations of the form (1). The discrete computational method shall have the form,

$$A^{(0)}\mathbf{Y}_m^{(i)} = \sum_{i=0}^{1} h^i e_i y_n^{(i)} + h^2 d_i f(y_n) + h^2 b_i f(\mathbf{Y}_m),\ i=0,1 \tag{4}$$

We shall seek the approximate solution to equation (1) in the integration interval $[x_n, x_{n+1}]$. We assume that the solution on the interval $[x_n, x_{n+1}]$ is locally approximated by the basis function,

$$y(x) = \sum_{j=0}^{r+s-1} \tau_j x^j \tag{5}$$

where τ_j are the real coefficients to be determined, s is the number of interpolation points, r is the number of collocation points and $h = x_n - x_{n-1}$ is a constant step-size of the partition of the interval $[\alpha, \beta]$ which is given by $\alpha = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = \beta$.

The polynomial (5) is assumed to pass through the interpolation points $(x_{n+s}, y_{n+s}), s = \frac{3}{5}, \frac{4}{5}$ and the collocation points $(x_{n+r}, f_{n+r}), r = 0\left(\frac{1}{5}\right)1$. This gives the following $(r+s)$ system of equations,

$$\left.\begin{aligned} &\sum_{j=0}^{r+s-1} \tau_j x^j = y_{n+s},\ s = \frac{3}{5}, \frac{4}{5} \\ &\sum_{j=0}^{r+s-1} j(j-1)\tau_j x^{j-2} = f_{n+r},\ r = 0\left(\frac{1}{5}\right)1 \end{aligned}\right\} \tag{6}$$

The $(r+s)$ undetermined coefficients τ_j are obtained by solving the system of nonlinear equations (6) using Gauss elimination method. This gives a continuous hybrid linear multistep method of the form;

$$y(x) = \alpha_{\frac{3}{5}}(t) y_{n+\frac{3}{5}} + \alpha_{\frac{4}{5}}(t) y_{n+\frac{4}{5}} + h^2 \left(\sum_{j=0}^{1} \beta_j(t) f_{n+j} + \beta_k(t) f_{n+k} \right),\ k = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5} \tag{7}$$

where the coefficients $\alpha_{\frac{3}{5}}, \alpha_{\frac{4}{5}}, \beta_0, \beta_{\frac{1}{5}}, \beta_{\frac{2}{5}}, \beta_{\frac{3}{5}}, \beta_{\frac{4}{5}}, \beta_1$ are given by;

$$\left.\begin{aligned}
&\alpha_{\frac{3}{5}} = 4 - 5t \\
&\alpha_{\frac{4}{5}} = 5t - 3 \\
&\beta_0 = -\frac{1}{252000}\left(156250t^7 - 656250t^6 + 1115625t^5 - 984375t^4 + 479500t^3 - 126000t^2 + 15880t - 672\right) \\
&\beta_{\frac{1}{5}} = \frac{1}{252000}\left(781250t^7 - 3062500t^6 + 4659375t^5 - 3368750t^4 + 1050000t^3 - 70295t + 10668\right) \\
&\beta_{\frac{2}{5}} = -\frac{1}{126000}\left(781250t^7 - 2843750t^6 + 3871875t^5 - 2340625t^4 + 525000t^3 + 15700t - 9744\right) \\
&\beta_{\frac{3}{5}} = \frac{1}{126000}\left(781250t^7 - 2625000t^6 + 3215625t^5 - 1706250t^4 + 350000t^3 - 29065t + 13524\right) \\
&\beta_{\frac{4}{5}} = -\frac{1}{252000}\left(781250t^7 - 2406250t^6 + 2690625t^5 - 1334375t^4 + 262500t^3 + 160t - 2688\right) \\
&\beta_1 = \frac{1}{252000}\left(156250t^7 - 437500t^6 + 459375t^5 - 218750t^4 + 42000t^3 - 535t - 84\right)
\end{aligned}\right\} \tag{8}$$

where $t = \dfrac{x - x_n}{h}$.

The continuous method (7) is then solved for the independent solution at the grid points to give the continuous block method:

$$y(t) = \sum_{j=0}^{1} \frac{(jh)^{(m)}}{m!} y_n^{(m)} + h^2 \left(\sum_{j=0}^{1} \sigma_j(t) f_{n+j} + \sigma_k f_{n+k} \right), \; k = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5} \tag{9}$$

where the coefficients $\sigma_i,\ i = 0\left(\frac{1}{5}\right)1$ are given by;

$$\left.\begin{aligned}
\sigma_0 &= -\frac{1}{2016}\left(1250t^7 - 5250t^6 + 8925t^5 - 7875t^4 + 3836t^3 - 1008t^2\right)\\
\sigma_{\frac{1}{5}} &= \frac{25}{2016}\left(250t^7 - 980t^6 + 1491t^5 - 1078t^4 + 336t^3\right)\\
\sigma_{\frac{2}{5}} &= -\frac{25}{1008}\left(250t^7 - 910t^6 + 1239t^5 - 749t^4 + 168t^3\right)\\
\sigma_{\frac{3}{5}} &= \frac{25}{1008}\left(250t^7 - 840t^6 + 1029t^5 - 546t^4 + 112t^3\right)\\
\sigma_{\frac{4}{5}} &= -\frac{25}{2016}\left(250t^7 - 770t^6 + 861t^5 - 427t^4 + 84t^3\right)\\
\sigma_1 &= \frac{1}{2016}\left(1250t^7 - 3500t^6 + 3675t^5 - 1750t^4 + 336t^3\right)
\end{aligned}\right\} \tag{10}$$

We then evaluate (9) at $t = \frac{1}{5}\left(\frac{1}{5}\right)1$ to give the one-step computational method of the form (4) where,

$$\mathbf{Y}_m = \left[y_{n+\frac{1}{5}} \; y_{n+\frac{2}{5}} \; y_{n+\frac{3}{5}} \; y_{n+\frac{4}{5}} \; y_{n+1} \right]^T, \; f(\mathbf{Y}_m) = \left[f_{n+\frac{1}{5}} \; f_{n+\frac{2}{5}} \; f_{n+\frac{3}{5}} \; f_{n+\frac{4}{5}} \; f_{n+1} \right]^T$$

$$\mathbf{y}_n^{(i)} = \left[y_{n-1}^{(i)} \; y_{n-2}^{(i)} \; y_{n-3}^{(i)} \; y_{n-4}^{(i)} \; y_n^{(i)} \right]^T, \; f(\mathbf{y}_n) = \left[f_{n-1} \; f_{n-2} \; f_{n-3} \; f_{n-4} \; f_n \right]^T$$

and $A^{(0)} = 5 \times 5$ identity matrix.

For $i = 0$:

$$e_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \; e_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{5} \\ 0 & 0 & 0 & 0 & \frac{2}{5} \\ 0 & 0 & 0 & 0 & \frac{3}{5} \\ 0 & 0 & 0 & 0 & \frac{4}{5} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \; d_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1231}{126000} \\ 0 & 0 & 0 & 0 & \frac{71}{3150} \\ 0 & 0 & 0 & 0 & \frac{123}{3500} \\ 0 & 0 & 0 & 0 & \frac{376}{7875} \\ 0 & 0 & 0 & 0 & \frac{61}{1008} \end{bmatrix}, \; b_0 = \begin{bmatrix} \frac{863}{50400} & \frac{-761}{63000} & \frac{941}{126000} & \frac{-341}{126000} & \frac{107}{25200} \\ \frac{544}{7875} & \frac{-37}{1575} & \frac{136}{7875} & \frac{-101}{15750} & \frac{8}{7875} \\ \frac{3501}{28000} & \frac{-9}{3500} & \frac{87}{2880} & \frac{-9}{875} & \frac{9}{5600} \\ \frac{1424}{7875} & \frac{176}{7875} & \frac{608}{7875} & \frac{-16}{1575} & \frac{16}{7875} \\ \frac{475}{2016} & \frac{25}{504} & \frac{125}{1008} & \frac{25}{1008} & \frac{11}{2016} \end{bmatrix}$$

For $i=1$:

$$e_1=\begin{bmatrix}0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\end{bmatrix},\quad d_1=\begin{bmatrix}0&0&0&0&\frac{19}{288}\\0&0&0&0&\frac{14}{225}\\0&0&0&0&\frac{51}{800}\\0&0&0&0&\frac{14}{225}\\0&0&0&0&\frac{19}{288}\end{bmatrix},\quad b_1=\begin{bmatrix}\frac{1427}{7200}&\frac{-133}{1200}&\frac{241}{3600}&\frac{-173}{7200}&\frac{3}{800}\\\frac{43}{150}&\frac{7}{255}&\frac{7}{255}&\frac{-1}{75}&\frac{1}{450}\\\frac{219}{800}&\frac{57}{400}&\frac{57}{400}&\frac{-21}{800}&\frac{3}{800}\\\frac{64}{255}&\frac{8}{75}&\frac{64}{255}&\frac{14}{255}&0\\\frac{25}{96}&\frac{25}{144}&\frac{25}{144}&\frac{25}{96}&\frac{19}{288}\end{bmatrix}$$

3. ANALYSIS OF BASIC PROPERTIES OF THE COMPUTATIONAL METHOD

Some basic properties of the computational method derived shall be discussed in this section.

3.1 Order of Accuracy and Error Constant of the Method

According to [17], the computational method (4) is said to be of uniform accurate order p, if p is the largest positive integer for which $\bar{c}_0=\bar{c}_1=\bar{c}_2=\ldots=\bar{c}_p=\bar{c}_{p+1}=0,\ \bar{c}_{p+2}\neq\bar{0}$. $\bar{c}_{p+2}$ is called the error constant and the local truncation error of the method is given by;

$$\bar{t}_{n+k}=\bar{c}_{p+2}h^{(p+2)}y^{(p+2)}(t)+O\left(h^{(p+3)}\right) \tag{11}$$

Therefore, for the computational method derived $\bar{c}_0=\bar{c}_1=\bar{c}_2=\bar{c}_3=\bar{c}_4=\bar{c}_5=\bar{c}_6=\bar{c}_7=\bar{0}$, implying that the order $p=\begin{bmatrix}6&6&6&6&6\end{bmatrix}^T$ and the error constant is give by

$$\bar{c}_8=\begin{bmatrix}-\frac{199}{9450000000}&-\frac{19}{369140625}&-\frac{141}{1750000000}&-\frac{8}{73828125}&-\frac{11}{75600000}\end{bmatrix}^T.$$

3.2 Consistency of the Method

The computational method (4) is consistent since it has order $p=6\geq 1$. Consistency controls the magnitude of the local truncation error committed at each stage of the computation, [18].

3.3 Zero-stability of the Method

Definition 3.1 [18]: The computational method (4) is said to be zero-stable, if the roots $z_{s,}\, s=1,2,\ldots,k$ of the first characteristic polynomial $\rho(z)$ defined by $\rho(z)=\det(zA^{(0)}-e_0)$ satisfies $|z_s|\leq 1$ and every root satisfying $|z_s|=1$ have multiplicity not exceeding the order of the differential equation. Moreover, as $h\to 0, \rho(z)=z^{r-\mu}(z-1)^{\mu}$, where μ is the order of the matrices $A^{(0)}$ *and* e_0.

For our method,

$$\rho(z)=\left|z\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}-\begin{bmatrix}0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\end{bmatrix}\right|=z^4(z-1)=0 \tag{12}$$

Therefore, $z_1 = z_2 = z_3 = z_4 = 0,\ z_5 = 1$. Hence, the computational method is zero-stable. Zero-stability controls the propagation of the error as the integration progresses.

3.4 Convergence of the Method

The computational method is convergent since it is consistent and zero-stable.

Theorem 3.1 [19]

A linear multistep method is convergent if and only if it is stable and consistent.

3.5 Region of Absolute Stability of the Method

Definition 3.2 [20]

Region of absolute stability is a region in the complex z plane, where $z = \lambda h$. It is defined as those values of z such that the numerical solutions of $y'' = -\lambda y$ satisfy $y_j \to 0$ as $j \to \infty$ for any initial condition.

Applying the boundary locus method, we obtain the stability polynomial for the computational method derives as;

$$\begin{aligned}\bar{h}(w) = &-h^{10}\left(\frac{1}{1230468750}w^5 + \frac{149}{14\,765625000}w^4\right) - h^8\left(\frac{1481}{29\,531250000}w^5 + \frac{893603}{177\,187500000}w^4\right) \\ &-h^6\left(\frac{311}{236250000}w^5 + \frac{42407}{59062500}w^4\right) - h^4\left(\frac{139}{3750}w^4 - \frac{1}{5000}w^5\right) - h^2\left(\frac{1}{50}w^5 + \frac{47}{75}w^4\right) \\ &+ w^5 - 2w^4\end{aligned} \tag{13}$$

The stability region for the computational method is shown in Fig. 3.1.

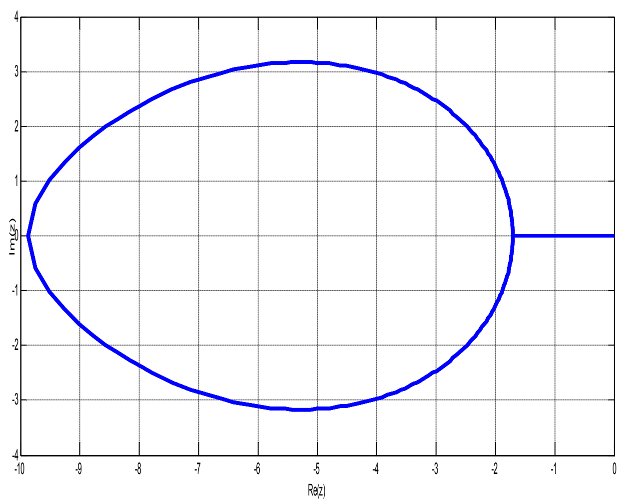

Fig. 3.1. Stability region of the computational method

The stability region in the Fig. 3.1 is A-stable.

4. RESULTS

4.1 Numerical Experiments

We shall apply the computational method derived in this research to simulate some Duffing oscillators that find applications in science and engineering.

The following notations shall be used in the tables below.

ESS-End point absolute errors obtained in [16]

EOM-Absolute error in [21]

EJS-Absolute error in [1]

EMU-Absolute error in [22]

ETG-Absolute error in [10]

Problem 4.1:

Consider the undamped Duffing equation,

$$y''(t)+y(t)+y^3(t)=\left(\cos t+\varepsilon\sin 10t\right)^3-99\varepsilon\sin 10t \quad (14)$$

with the initial conditions,

$$y(0)=1,\ y'(0)=10\varepsilon \quad (15)$$

where $\varepsilon=10^{-10}$. The exact solution is given by,

$$y(t)=\cos t+\varepsilon\sin 10t \quad (16)$$

This equation describes a periodic motion of low frequency with a small perturbation of high frequency.

Source: [21]

Problem 4.2:

Consider the following undamped Duffing equation of the form;

$$y''(t)+y(t)+y^3(t)=B\cos\Omega t \quad (17)$$

with initial conditions,

$$y(0)=\alpha,\ y'(0)=0 \quad (18)$$

where,

$$\alpha=0.200426728067, B=0.002,\ \Omega=1.01$$

The exact solution to the problem is

$$y(t)=\sum_{i=0}^{3}A_{2i+1}Cos\left(\left(2i+1\right)\Omega t\right) \quad (19)$$

where,

$$\begin{Bmatrix}A_1,A_3,A_5,\\ A_7,A_9\end{Bmatrix}=\begin{Bmatrix}0.200179477536, 0.0024946143, 0.000000304014,\\ 0.000000000374, 0.000000000000\end{Bmatrix}$$

Source: [16]

Problem 4.3:

Consider the damped Duffing equation,

$$y''(t)+2y'(t)+y(t)+8y^3(t)=e^{-3t} \quad (20)$$

with the initial conditions,

$$y(0)=\frac{1}{2},\ y'(0)=-\frac{1}{2} \quad (21)$$

The exact solution is given by,

$$y(t)=\frac{1}{2}e^{-t} \quad (22)$$

Source: [22]

Problem 4.4:

Consider the damped Duffing equation,

$$y''(t)+y'(t)+y(t)+y^3(t)=\cos^3(t)-\sin(t) \quad (23)$$

whose initial conditions are,

$$y(0)=1,\ y'(0)=0 \quad (24)$$

The exact solution is given by,

$$y(t)=\cos(t) \quad (25)$$

Source: [10]

Problem 4.5:

Consider the undamped Duffing equation,

$$y''(t)+3y(t)+2y^3(t)=\cos(t)\sin(2t) \quad (26)$$

with the initial conditions,

$$y(0)=0,\ y'(0)=1 \quad (27)$$

The exact solution is given by,

$$y(t)=\sin(t) \quad (28)$$

Source: [12]

Table 4.1. Showing the results for problem 5.1 in comparison with the absolute errors in [21]

t	Exact solution	Computed solution	Error	EOM	Time/s
0.0025	0.9999968750041274	0.9999968750041274	0.000000e+000	0.000000e+000	0.1039
0.0050	0.9999875000310395	0.9999875000310395	0.000000e+000	1.110223e-016	0.1348
0.0075	0.9999718751393287	0.9999718751393286	1.110223e-016	8.881784e-016	0.1736
0.0100	0.9999500004266486	0.9999500004266486	0.000000e+000	7.771561e-016	0.2112
0.0125	0.9999218760297148	0.9999218760297148	0.000000e+000	4.440892e-016	0.2121
0.0150	0.9998875021243030	0.9998875021243031	1.110223e-016	9.992007e-016	0.2127
0.0175	0.9998468789252486	0.9998468789252487	1.110223e-016	1.665335e-015	0.2133
0.0200	0.9998000066864446	0.9998000066864449	2.220446e-016	2.775558e-015	0.2140
0.0225	0.9997468857008414	0.9997468857008415	1.110223e-016	5.440093e-015	0.2146
0.0250	0.9996875163004431	0.9996875163004431	0.000000e+000	7.216450e-015	0.2152
0.0275	0.9996218988563066	0.9996218988563066	0.000000e+000	9.436896e-015	0.2160

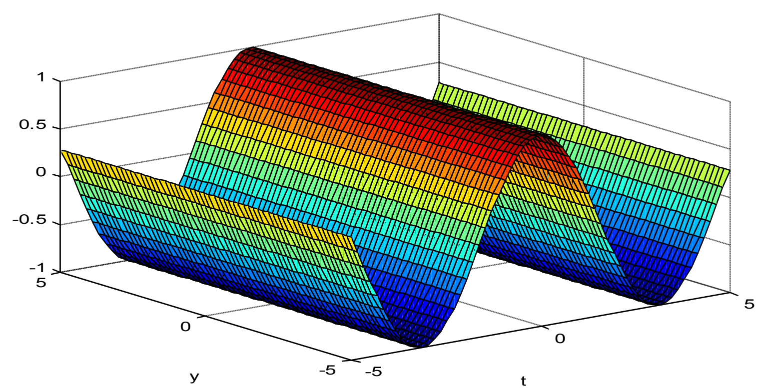

Fig. 4.1. Graphical result showing the oscillatory nature of problem 4.1

Table 4.2. Comparison of the end-point absolute errors in [1] and [16] with that of the new method

h	Error	EJS	ESS
$\frac{M}{500}$	4.846124e-015	8.813783e-013	1.81e-010
$\frac{M}{1000}$	2.148108e-014	1.114692e-012	8.02e-012
$\frac{M}{2000}$	9.221651e-014	2.953554e-012	5.52e-012
$\frac{M}{3000}$	2.008060e-014	2.339406e-012	7.28e-012
$\frac{M}{4000}$	2.930989e-014	1.859929e-012	6.99e-012
$\frac{M}{5000}$	3.613776e-014	1.328992e-012	6.65e-012

Note: $M=10$ *in Table 4.1 above*

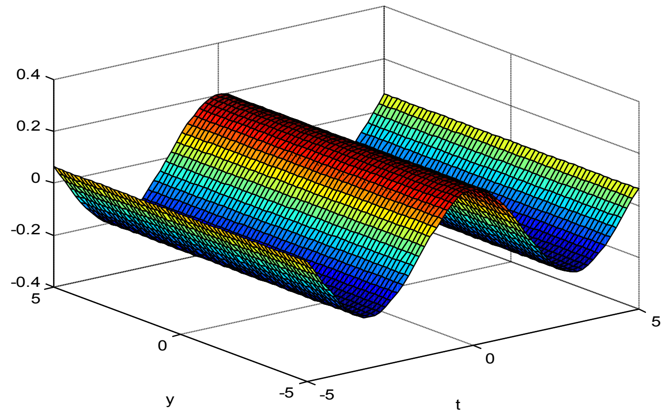

Fig. 4.2. Graphical result showing the oscillatory nature of problem 4.2

Table 4.3. Showing the results for problem 4.3 in comparison with the absolute errors in [22]

t	Exact solution	Computed solution	Error	EMU	Time/s
0.1000	0.4524187090179798	0.4524187090179798	0.000000e+000	1.487e-08	0.0411
0.2000	0.4093653765389909	0.4093653765389909	0.000000e+000	1.286e-07	0.0474
0.3000	0.3704091103408589	0.3704091103408589	0.000000e+000	1.464e-07	0.0539
0.4000	0.3351600230178196	0.3351600230178196	0.000000e+000	1.393e-07	0.0603
0.5000	0.3032653298563167	0.3032653298563167	0.000000e+000	1.845e-07	0.0669
0.6000	0.2744058180470131	0.2744058180470131	0.000000e+000	2.422e-07	0.0735
0.7000	0.2482926518957047	0.2482926518957047	0.000000e+000	2.468e-07	0.0799
0.8000	0.2246644820586107	0.2246644820586106	2.775558e-017	2.127e-07	0.0866
0.9000	0.2032848298702994	0.2032848298702994	0.000000e+000	1.987e-07	0.0929
1.0000	0.1839397205857211	0.1839397205857210	5.551115e-017	2.071e-07	0.0998

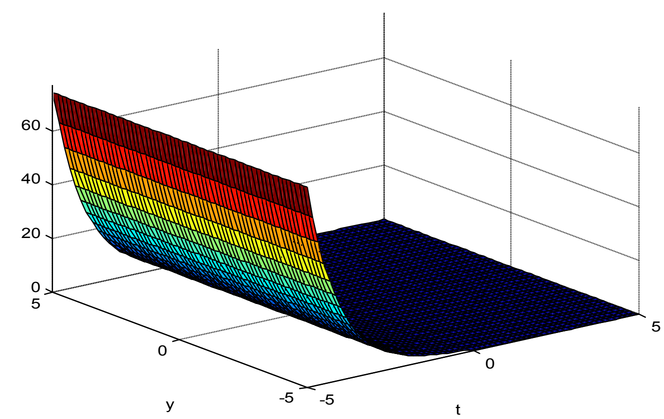

Fig. 4.3. Graphical result showing the oscillatory nature of problem 4.3

Table 4.4. Showing the results for problem 4.4 in comparison with the absolute errors in [10]

t	Exact solution	Computed solution	Error	ETG	Time/s
0.1000	0.9950041652780258	0.9950041652780257	1.110223e-016	9.418022e-013	0.0093
0.2000	0.9800665778412416	0.9800665778412414	2.220446e-016	9.320766e-012	0.0160
0.3000	0.9553364891256060	0.9553364891256060	0.000000e+000	2.371603e-011	0.0234
0.4000	0.9210609940028850	0.9210609940028852	2.220446e-016	4.248379e-011	0.0301

t	Exact solution	Computed solution	Error	ETG	Time/s
0.5000	0.8775825618903727	0.8775825618903725	1.110223e-016	6.390422e-011	0.0367
0.6000	0.8253356149096781	0.8253356149096780	1.110223e-016	8.632239e-011	0.0434
0.7000	0.7648421872844882	0.7648421872844881	1.110223e-016	1.082653e-010	0.0500
0.8000	0.6967067093471651	0.6967067093471649	1.110223e-016	1.285219e-010	0.0567
0.9000	0.6216099682706640	0.6216099682706638	1.110223e-016	1.461836e-010	0.0634
1.0000	0.5403023058681392	0.5403023058681390	2.220446e-016	1.606468e-010	0.0704

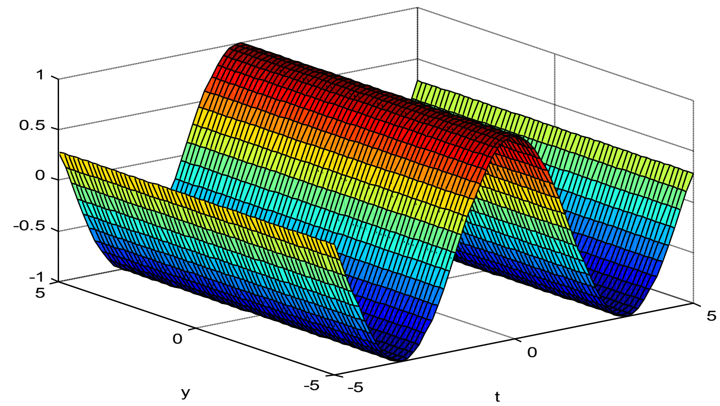

Fig. 4.4. Graphical result showing the oscillatory nature of problem 4.4

Table 4.5. Showing the results for problem 4.5 in comparison with the absolute errors in [12]

t	Exact solution	Computed solution	Error	EJS	Time/s
0.1000	0.0998334166468281	0.0998334166468282	1.387779e-017	3.024248e-013	0.0437
0.2000	0.1986693307950612	0.1986693307950612	0.000000e+000	4.584944e-013	0.0492
0.3000	0.2955202066613397	0.2955202066613396	1.110223e-016	7.316370e-014	0.0547
0.4000	0.3894183423086507	0.3894183423086505	2.220446e-016	1.692257e-012	0.0603
0.5000	0.4794255386042032	0.4794255386042029	2.775558e-016	4.596878e-012	0.0662
0.6000	0.5646424733950356	0.5646424733950353	3.330669e-016	8.754997e-012	0.0719
0.7000	0.6442176872376914	0.6442176872376908	5.551115e-016	1.390665e-011	0.0775
0.8000	0.7173560908995231	0.7173560908995226	5.551115e-016	1.959244e-011	0.0831
0.9000	0.7833269096274838	0.7833269096274829	8.881784e-016	2.519718e-011	0.0888
1.0000	0.8414709848078968	0.8414709848078962	6.661338e-016	2.999911e-011	0.0946

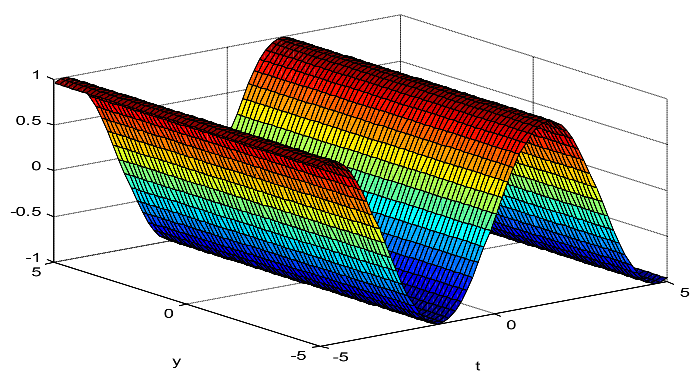

Fig. 4.5. Graphical result showing the oscillatory nature of problem 4.5

5. DISCUSSION OF RESULTS

We simulated some Duffing oscillators with the aid of the computational method developed and from the results obtained, it is obvious that the computational method developed is more efficient than the existing ones with which we compared our results.

6. CONCLUSION

A one-step computational method has been developed for the simulation of Duffing oscillators using the power series approximate solution. It is obvious from the results (numerical and graphical) obtained that the method is computationally reliable. The method developed was also found to be consistent, convergent, zero-stable and A-stable. This paper therefore recommends the use of this method for solving not only Duffing equations but second order nonlinear (and linear) differential equations of the form (1).

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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