British Journal of Mathematics & Computer Science
19(2): 1-15, 2016, Article no.BJMCS.28691
ISSN: 2231-0851

SCIENCEDOMAIN *international*
www.sciencedomain.org

Iterative Approximation of Solutions of Hammerstein Integral Equations with Maximal Monotone Operators in Banach Spaces

M. O. Uba[1*], M. A. Onyido[1] and P. U. Nwokoro[1]

[1] *Department of Mathematics, University of Nigeria, Nsukka, Nigeria.*

Authors' contributions

This work was carried out in collaboration between all authors. All authors made significant contribution. Author MOU wrote the first draft of the manuscript. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/BJMCS/2016/28691
Editor(s):
(1) Wei-Shih Du, Department of Mathematics, National Kaohsiung Normal University, Taiwan.
Reviewers:
(1) Abdullah Sonmezoglu, Bozok University, Turkey.
(2) Octav Olteanu, University Politehnica of Bucharest, Romania.
Complete Peer review History: http://www.sciencedomain.org/review-history/16555

Received: 31st July 2016
Accepted: 9th August 2016
Published: 14th October 2016

Original Research Article

Abstract

Let X be a uniformly convex and uniformly smooth real Banach space with dual space X^*. Let $F : X \to X^*$ and $K : X^* \to X$ be bounded maximal monotone mappings. Suppose the Hammerstein equation $u + KFu = 0$ has a solution. An iteration sequence is constructed and proved to converge strongly to a solution of this equation.

Keywords: Bounded maximal monotone mappings; hammerstein equations; strong convergence.

2010 Mathematics Subject Classification: 47H05, 47J25, 45L05.

*Corresponding author: E-mail: markjoe.uba@unn.edu.ng;

1 Introduction

Let H be a real Hilbert space. A map $A : H \to 2^H$ is called *monotone* if for each $x, y \in H$, the following inequality holds:

$$\langle \xi - \tau, x - y \rangle \geq 0 \ \ \forall \ \xi \in Ax, \tau \in Ay. \tag{1.1}$$

In a case where A is single-valued, A is said to be *angle-bounded* with angle $\beta > 0$ if

$$\langle Ax - Ay, z - y \rangle \leq \beta \langle Ax - Ay, x - y \rangle \tag{1.2}$$

for any triple elements $x, y, z \in H$. For $y = z$ inequality (1.2) implies the monotonicity of A. A monotone *linear* operator $A : H \to H$ is said to be *angle bounded* with angle $\alpha > 0$ if

$$|\langle Ax, y \rangle - \langle Ay, x \rangle| \leq 2\alpha \langle Ax, x \rangle^{\frac{1}{2}} \langle Ay, y \rangle^{\frac{1}{2}} \tag{1.3}$$

for all $x, y \in H$.

Let E be a real normed space, E^* its topological dual space. A map $J : E \to 2^{E^*}$ defined by

$$Jx := \left\{ x^* \in E^* : \left\langle x, x^* \right\rangle = \|x\|.\|x^*\|, \ \|x\| = \|x^*\| \right\}$$

is called the *normalized duality map* on E.

A map $A : E \to 2^E$ is called *accretive* if for each $x, y \in E$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle \xi - \tau, j(x - y) \rangle \geq 0, \ \ \forall \ \xi \in Ax, \tau \in Ay. \tag{1.4}$$

A map $A : E \to 2^{E^*}$ is called *monotone* if for all $x, y \in D(A)$

$$\langle \xi - \zeta, x - y \rangle \geq 0 \ \ \forall \xi \in Ax, \ \forall \zeta \in Ay. \tag{1.5}$$

A mapping $A : E \to 2^{E^*}$ is said to be maximal monotone if it is monotone and for $(x, u) \in E \times E^*$ the inequalities $\langle u - v, x - y \rangle \geq 0$, for all $(y, v) \in G(A)$, imply $(x, u) \in G(A)$ where $G(A)$ is the graph of A.

In a Hilbert space, the normalized duality map is the identity map. Hence, in Hilbert spaces, *monotonicity* and *accretivity* coincide.

Monotone mappings were studied in Hilbert spaces by Zarantonello [1], Minty [2], Kačurovskii [3] and a host of other authors. Interest in such mappings stems mainly from their usefulness in numerous applications.

Example 1.1. *Consider the following: Let $f : H \to \mathbb{R} \cup \{\infty\}$ be a proper convex function. The subdifferential of f at $u \in H$ is defined by*

$$\partial f(u) = \left\{ u^* \in H : f(y) - f(u) \geq \left\langle y - u, u^* \right\rangle \ \ \forall \ y \in H \right\}.$$

It is easy to see that $\partial f : H \to 2^H$ is a monotone operator on H, and that $0 \in \partial f(u)$ if and only if u is a minimizer of f. Setting $\partial f \equiv A$, it follows that solving the inclusion $0 \in Au$, in this case, is solving for a minimizer of f.

In fact, we state the following remark made by Pascali and Sburian in [4].

> ...The monotone maps constitute the most manageable class, because of the very simple structure of the monotonicity condition. The monotone mappings appear in a rather wide variety of contexts, since they can be found in many functional equations. Many of them appear also in calculus of variations, as subdifferential of convex funtions (Pascali and Sburian [4], p. 101).

Let $\Omega \subset \mathbb{R}^n$ be bounded. Let $k : \Omega \times \Omega \to \mathbb{R}$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be measurable real-valued functions. An integral equation (generally nonlinear) of Hammerstein-type has the form

$$u(x) + \int_{\Omega} k(x,y) f(y, u(y)) dy = w(x), \tag{1.6}$$

where the unknown function u and inhomogeneous function w lie in a Banach space E of measurable real-valued function. If we define $F : \mathcal{F}(\Omega, \mathbb{R}) \to \mathcal{F}(\Omega, \mathbb{R})$ and $K : \mathcal{F}(\Omega, \mathbb{R}) \to \mathcal{F}(\Omega, \mathbb{R})$ by

$$Fu(y) = f(y, u(y)), \ x \in \Omega,$$

and

$$Kv(x) = \int_{\Omega} k(x,y) v(y) dy, \ x \in \Omega,$$

respectively, where $\mathcal{F}(\Omega, \mathbb{R})$ is a space of measurable real-valued functions defined from Ω to $\mathbb{R}$, then equation (1.6) can be put in an abstract form

$$u + KFu = w. \tag{1.7}$$

Without loss of generality we may assume that $w \equiv 0$ so that (1.7) becomes

$$u + KFu = 0. \tag{1.8}$$

Interest in (1.6) stems mainly from the fact that several problems that arise in differential equations, for instance, elliptic boundary value problems whose linear part posses Green's function can, as a rule be transformed into the form (1.6) (see e.g., Pascali and Sburian [4], chapter IV, p. 164. see also Chidume and Djitte [5, 6], Chidume and Yekini [7]).

Equations of Hammerstein-type also play a crucial role in the theory of optimal control systems and in automation and network theory (see e.g., Dolezale [8]).

Several existence results have been proved for equations of Hammerstein-type (see e.g., Brézis and Browder [9, 10, 11], Browder [12], Browder, De Figueiredo and Gupta [13]).

In general, equations of Hammerstein-type are nonlinear and there is no known method to find close form solutions for them. Consequently, methods of approximating solutions of such equations, where solutions are known to exist, are of interest. In the special case where *one of the operators in equation 1.7 is angle-bounded*, and the other is *bounded*, Brézis and Browder [9, 11] proved the strong convergence of a suitably defined *Galerkin approximation* to a solution of equation (1.7). In fact, they prove the following theorem.

Theorem 1.2 (Brézis and Browder [11])**.** *Let H be a seprable Hilbert space and C be a closed subspace of H. Let $K : H \to C$ be a bounded continuous monotone operator and $F : C \to H$ be angle-bounded and weakly compact mapping. For a given $f \in C$, consider the Hammerstein equation*

$$(I + KF)u = f \tag{1.9}$$

and its nth Galerking approximation given by

$$(I + K_n F_n)u_n = P^* f, \tag{1.10}$$

where $K_n = P_n^* K P_n : H \to C_n$ and $F_n = P_n F P_n^* : C_n \to H$, where the symbols have their usual meanings (see [4]). Then, for each $n \in \mathbb{N}$, the Galerkin approximation (1.10) admits a unique solution u_n in C_n and $\{u_n\}$ converges strongly in H to the unique solution $u \in C$ of the equation (1.9).

Remark 1.1. Theorem 1.2 is a special case of the actual theorem of Brézis and Browder in which the Banach space is a separable real Hilbert space. The main theorem of Brézis and Browder is proved in an arbitrary separable real Banach space.

We observe that the Galerkin method of Brézis and Browder is not iterative. Consequently, if an *iterative algorithm* can be developed for the approximation of solutions of equation of Hammerstein-type (1.7), this will certainly be a welcome complement to the Galerkin approximation method. Attempts had been made to approximate solutions of equations of Hammerstein-type using *Mann-type* (see e.g., Mann [14]) iteration scheme. However, the results obtained were not satisfactory (see [15]). The recurrence formulas used in these attempts, even in real Hilbert spaces, involved K^{-1} which is required to be strongly monotone when K is, and this, apart from limiting the class of mappings to which such iterative schemes are applicable, is also not convenient in any possible applications.

Part of the difficulty in establishing *iterative algorithms* for approximating solutions of Hammerstein equations seems to be that the composition of two monotone maps need not be monotone.

The first satisfactory results on *iterative methods* for approximating solutions of Hammerstein equations, as far as we know, were obtained by Chidume and Zegeye [16, 17, 18].

Let X be a real Banach space and $F, K : X \to X$ be *accretive-type* mappings. Let $E := X \times X$. Then, Chidume and Zegeye (see [16, 17]) defined $T : E \to E$ by

$$T[u, v] = [Fu - v, Kv + u] \text{ for } [u, v] \in E.$$

We note that $T[u, v] = 0$ if and only if u solves (1.8) and $v = Fu$. The authors obtained strong convergence theorems for solutions of Hammerstein equations under various continuity conditions in the cartesian product space E.

The method of proof used by Chidume and Zegeye provided the clue to the establishement of the following coupled explicit algorithm for computing a solution of the equation $u + KFu = 0$ in the original space X. With initial vectors $u_0, v_0 \in X$, sequences $\{u_n\}$ and $\{v_n\}$ in X are defined iteratively as follows:

$$u_{n+1} = u_n - \alpha_n(Fu_n - v_n),\ n \geq 0, \tag{1.11}$$

$$v_{n+1} = v_n - \alpha_n(Kv_n + u_n),\ n \geq 0, \tag{1.12}$$

where α_n is a sequence in $(0, 1)$ satisfying appropriate conditions. The recursion formulas (1.11) and (1.12) have been used successfully to approximate solutions of Hammerstein equations involving nonlinear accretive-type mappings. Following this, several authors have studied the recursion formulas (1.11), (1.12) (see e.g., Chidume and Djitte [5, 6], Chidume and Yekini [7]) and proved several strong convergence theorems.

Remark 1.2. Even though the class of monotone-type operators have a wider variety of applications than the class of accretive-type operators in Banach spaces, virtually all the results on the approximation of solutions of Hammerstein equations are either proved in Hilbert spaces or in a Banach space in the case where the operators K and F are *accretive-type* mappings (see [19], [20] and [7]). As far as we know, there are very few results on the approximation of solutions of Hammerstein-type equations in Banach spaces (in the case where the operators K and F are *monotone-type operators*).

Remark 1.3. It seems that, part of the difficulty is that since the operator F maps E to E^* and K maps E^* to E the recursion formulas used for accretive-type mappings may no longer make sense under these settings. Moreover, most of the inequalities used in proving convergence when the operators are accetive-type involve the normalized duality mappings which also appears in the definition of accretive operators. However, the definition of monotone mappings does not involve the normalized duality mappings. This creates computational difficulties in attempting to use standard Banach space inequalities in proving convergence results for *monotone-type* mappings.

Recently, the following important theorem was proved by Chidume and Ofoedu.

Theorem 1.3. *[Chidume and Ofoedu, [19]]Let E be a $2-$uniformly smooth real Banach space. Let $F, K : E \to E$ be bounded and accretive mappings. Let $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ be sequences in E defined iteratively from arbitrary points $u_1,\ v_1 \in E$ by*

$$u_{n+1} = u_n - \lambda_n\alpha_n(Fu_n - v_n) - \lambda_n\theta_n(u_n - u_1),\ n \geq 1, \tag{1.13}$$

$$v_{n+1} = v_n - \lambda_n\alpha_n(Kv_n + u_n) - \lambda_n\theta_n(v_n - v_1),\ n \geq 1, \tag{1.14}$$

where $\{\lambda_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\theta_n\}_{n=1}^{\infty}$ are real sequences in $(0,1)$ such that $\lambda_n = o(\theta_n)$, $\alpha_n = o(\theta_n)$, and $\sum_{n=1}^{\infty}\lambda_n\theta_n = \infty$. Suppose that $u + KFu = 0$ has a solution $u^* \in E$. Then, there exist real constants $\varepsilon_0,\ \varepsilon_1 > 0$ and a set $\Omega \subset W = E \times E$ such that if $\lambda_n \leq \varepsilon_0\theta_n$ and $\alpha_n \leq \varepsilon_1\theta_n$, $\forall\ n \geq n_0$, for some $n_0 \in \mathbb{N}$ and $w^* := (u^*, v^*) \in \Omega$ (where $v^* = Fu^*$), the sequence $\{u_n\}_{n=1}^{\infty}$ converges strongly to u^*.

It is our purpose in this paper to construct an iteration sequence and prove its strong convergence to a solution of $u + KFu = 0$ in uniformly convex and uniformly smooth real Banach spaces. Furthermore, our result complements Theorem 1.3 to provide iterative methods for the approximation of solutions of the Hammerstein equation $u + KFu = 0$ in more general spaces when the operators K and F are *bounded maximal monotone-type operators.* Our method of proof is different and of independent interest.

2 Preliminaries

Let E be a normed space with $dimE \geq 2$. The *modulus of convexity* of E is the function $\delta_E : (0,2] \to [0,1]$, defined by

$$\delta_E(\epsilon) := inf\Big\{1 - \Big\|\frac{x+y}{2}\Big\| : \|x\| = \|y\| = 1; \epsilon = \|x - y\|\Big\}.$$

The space E is *uniformly convex* if and only if $\delta_E(\epsilon) > 0$ for every $\epsilon \in (0,2]$.

It is also well known (see *e.g.*, Chidume [21] p. 34, also Lindenstrauss and Tzafriri [22]) that δ_E is nondecreasing. If there exist a constant $c > 0$ and a real number $p > 1$ such that $\delta_E(\epsilon) \geq c\epsilon^p$, then E is said to be *p-uniformly convex.* Typical examples of such spaces are the L_p, ℓ_p and W_p^m spaces for $1 < p < \infty$ where,

$$L_p\ (or\ l_p)\ or\ W_p^m \text{ is } \begin{cases} p-\text{uniformly convex} & \text{if } 2 \leq p < \infty; \\ 2-\text{uniformly convex} & \text{if } 1 < p < 2. \end{cases}$$

A Banach space E is said to be *strictly convex* if

$$\|x\| = \|y\| = 1, \quad x \neq y \implies \Big\|\frac{x+y}{2}\Big\| < 1.$$

Let E be a real normed linear space of dimension ≥ 2. The *modulus of smoothness* of E , $\rho_E : [0,\infty) \to [0,\infty)$, is defined by:

$$\rho_E(\tau) := \sup\Big\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau, \quad \tau > 0\Big\}.$$

A normed linear space E is called *uniformly smooth* if

$$\lim_{\tau \to 0} \frac{\rho_E(\tau)}{\tau} = 0.$$

The norm of E is said to be *Fréchet differentiable* if for each $x \in S := \{u \in E : \|u\| = 1\}$,

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t},$$

exists and is attained uniformly for $y \in E$.

Let E be a smooth real Banach space with dual E^*. The function $\phi : E \times E \to \mathbb{R}$, is defined by,

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \text{for } x, y \in E, \tag{2.1}$$

where J is the normalized duality mapping from E into 2^{E^*}. It was introduced by Alber and has been studied by Alber [23], Alber and Guerre-Delabriere [24], Kamimura and Takahashi [25], Reich [26] and a host of other authors. If $E = H$, a real Hilbert space, then equation (2.1) reduces to $\phi(x, y) = \|x - y\|^2$ for $x, y \in H$. It is obvious from the definition of the function ϕ that

$$(\|x\| - \|y\|)^2 \le \phi(x, y) \le (\|x\| + \|y\|)^2 \quad \text{for } x, y \in E. \tag{2.2}$$

Define a map $V : X \times X^* \to \mathbb{R}$ by

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2. \tag{2.3}$$

Then, it follows that

$$V(x, x^*) = \phi(x, J^{-1}(x^*)) \;\forall\; x \in X, \; x^* \in X^*. \tag{2.4}$$

Lemma 2.1. *([Alber, [23]]) Let X be a reflexive striclty convex and smooth Banach space with X^* as its dual. Then,*

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \le V(x, x^* + y^*) \tag{2.5}$$

for all $x \in X$ and $x^*, y^* \in X^*$.

Lemma 2.2 (Kamimura and Takahashi, [25])**.** *Let X be a real smooth and uniformly convex Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences of X. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \to 0$ as $n \to \infty$, then $\|x_n - y_n\| \to 0$ as $n \to \infty$.*

Lemma 2.3. *(Xu [27]) Let ρ_n be a sequence of non-negative real numbers satisfying the relation:*

$$\rho_{n+1} \le (1 - \beta_n)\rho_n + \beta_n \zeta_n + \gamma_n, \; n \ge 0, \tag{2.6}$$

where,
(i) $\beta_n \subset [0, 1]$, $\sum \beta_n = \infty$; (ii) $\limsup \zeta_n \le 0$; (iii) $\gamma_n \ge 0$; $(n \ge 0)$, $\sum \gamma_n < \infty$. Then, $\rho_n \to 0$ as $n \to \infty$.

Remark 2.1. Let E^* be a strictly convex dual Banach space with a Fréchet differentiable norm and $A : E \to 2^{E^*}$, be a maximal monotone map with no monotone extension. Let $z \in E^*$ be fixed. Then for every $\lambda > 0$, there exists a unique $x_\lambda \in E$ such that $z \in Jx_\lambda + \lambda A x_\lambda$ (see Reich [28], p. 342). Setting $J_\lambda z = x_\lambda$, we have the resolvent $J_\lambda := (J + \lambda A)^{-1} : E^* \to E$ of A, for every $\lambda > 0$. A celebrated result of Reich follows.

Lemma 2.4. *(Reich, [28]). Let E^* be a strictly convex dual Banach space with a Fréchet differentiable norm and let $A : E \to E^*$ be maximal monotone such that $A^{-1}0 \neq \emptyset$. Let $z \in E^*$ be an arbitrary but fixed vector. For each $\lambda > 0$, there exists a unique $x_\lambda \in E$ such that $z \in Jx_\lambda + \lambda Ax_\lambda$. Furthermore, x_λ converges strongly to a unique $v \in A^{-1}0$.*

Lemma 2.5 (Alber, [29]). *Let X be a uniformly convex Banach space. Then for any $R > 0$ and any x, $y \in X$ such that $\|x\| \leq R$, $\|y\| \leq R$ the following inequality holds:*

$$\langle Jx - Jy, x - y\rangle \geq (2L)^{-1}\delta_X(c_2^{-1}\|x - y\|), \tag{2.7}$$

where $c_2 = 2\max\{1, R\}$, $1 < L < 1.7$.

Define

$$K := 4RLsup\{\|Jx - Jy\| : \|x\| \leq R, \|y\| \leq R\} + 1 \tag{2.8}$$

Lemma 2.6 (Alber, [29]). *Let X be a uniformly smooth and strictly convex Banach space. Then for any $R > 0$ and any x, $y \in X$ such that $\|x\| \leq R$, $\|y\| \leq R$ the following inequality holds:*

$$\langle Jx - Jy, x - y\rangle \geq (2L)^{-1}\delta_{X^*}(c_2^{-1}\|Jx - Jy\|), \tag{2.9}$$

where $c_2 = 2\max\{1, R\}$, $1 < L < 1.7$.

Lemma 2.7 (Alber, [29]). *Let X be a reflexive strictly convex and smooth Banach space with dual X^*. Let $W : X \times X \to \mathbb{R}$ be defined by $W(x, y) = \frac{1}{2}\phi(y, x)$. Then,*

$$\phi(y, x) - \phi(y, z) \geq 2\langle Jx - Jz, z - y\rangle, \tag{2.10}$$

and

$$W(x, y) \leq \langle Jx - Jy, x - y\rangle, \tag{2.11}$$

for all $x, y, z \in X$

Lemma 2.8 (Chidume et al., [30]). *From Lemma 2.4, setting $\lambda_n := \frac{1}{\theta_n}$ where $\theta_n \to 0$ as $n \to \infty$, $\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}K\right) \leq 1$, $z = Jv$ for some $v \in E$, and $y_n := \left(J + \frac{1}{\theta_n}A\right)^{-1} z$, we obtain that:*

$$Ay_n = \theta_n(Jv - Jy_n), \tag{2.12}$$

$$y_n \to y^* \in A^{-1}0,$$

where K is as in lemma 2.5 and $A : E \to E^*$ is maximal monotone. We observe that equation (2.12) yields

$$Jy_{n-1} - Jy_n + \frac{1}{\theta_n}\Big(Ay_{n-1} - Ay_n\Big) = \frac{\theta_{n-1} - \theta_n}{\theta_n}\Big(Ju - Jy_{n-1}\Big).$$

Taking the duality pairing of this with $y_{n-1} - y_n$ and using monotonicity of A, we obtain that

$$\langle Jy_{n-1} - Jy_n, y_{n-1} - y_n\rangle \leq \frac{\theta_{n-1} - \theta_n}{\theta_n}\Big\|Ju - Jy_{n-1}\Big\|\|y_{n-1} - y_n\|.$$

We observe that if E is uniformly convex and uniformly smooth, using lemma 2.5 we obtain,

$$(2L)^{-1}\delta_E(c_2^{-1}\|y_{n-1} - y_n\|) \leq \frac{\theta_{n-1} - \theta_n}{\theta_n}\Big\|Ju - Jy_{n-1}\Big\|\|y_{n-1} - y_n\|,$$

which gives

$$\|y_{n-1} - y_n\| \leq c_2\delta_E^{-1}\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}K\right),\ \textit{for some } K > 0. \tag{2.13}$$

Similarly, using equation 2.9 of lemma 2.6, we obtain that,

$$\|Jy_{n-1} - Jy_n\| \leq c_2\delta_{E^*}^{-1}\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}K\right),\textit{for some } K > 0. \tag{2.14}$$

The following important results are known.

Lemma 2.9. *Let E be a smooth real Banach space with dual E^* and the function $\phi : E \times E \to \mathbb{R}$ defined by,*

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \textit{for } x, y \in E,$$

where J is the normalized duality mapping from E into 2^{E^*}. Then,

$$\phi(y, x) = \phi(x, y) + 2\langle x, Jy\rangle - 2\langle y, Jx\rangle. \tag{2.15}$$

Lemma 2.10. *Let X, X^* be uniformly convex and uniformly smooth real Banach spaces. Let $E = X \times X^*$ with the norm $\|z\|_E = (\|u\|_X + \|v\|_{X^*})^{\frac{1}{2}}$, for any $z = [u, v] \in E$. Let $E^* = X^* \times X$ denote the dual space of E. For arbitrary $x = [x_1, x_2] \in E$, define the map $J_E : E \to E^*$ by*

$$J_E(x) = J_E[x_1, x_2] := [J_X(x_1), J_{X^*}(x_2)],$$

so that for arbitrary $z_1 = [u_1, v_1]$, $z_2 = [u_2, v_2]$ in E, the duality pairing $\langle\cdot,\cdot\rangle$ is given by

$$\langle z_1, J_E\rangle := \langle u_1, J_X(u_2)\rangle + \langle v_1, J_{X^*}(v_2)\rangle.$$

Then, E is uniformly smooth and uniformly convex.

Lemma 2.11. *Let E be a uniformly convex and uniformly smooth real Banach and $F : E \to E^*$, $K : E^* \to E$ be maximal monotone. Define $A : E \times E^* \to E^* \times E$ by*

$$A[u, v] = [Fu - v, Kv + u]\ \forall\ [u, v] \in E \times E^*.$$

Then, A is maximal monotone.

Remark 2.2. From Lemma 2.4, setting $\lambda_n := \frac{\alpha_n}{\theta_n}$ where $\frac{\theta_n}{\alpha_n} \to 0$ as $n \to \infty$, $z = [z_1, z_2] = J_{E\times E^*}[u, v]$ for some $[u, v] \in E \times E^*$, and $[y_n, y_n^*] := \left(J_{E\times E^*} + \frac{\alpha_n}{\theta_n}A\right)^{-1}[z_1, z_2]$, we obtain that:

$$Jy_n + \frac{\alpha_n}{\theta_n}(Fy_n - y_n^*) = z_1,\ \forall n \geq 0,\ \textit{and} \tag{2.16}$$

$$J_*y_n^* + \frac{\alpha_n}{\theta_n}(Ky_n^* + y_n) = z_2\ \forall n \geq 0; \tag{2.17}$$

Remark 2.3. Let $y_n \to y$ and $y_n^* \to y^*$. From lemma 2.4 we have that $[y_n, y_n^*]$ converges to a point in $A^{-1}0$. This implies that $[y, y^*] \in A^{-1}0$. Consequently, $A[y, y^*] = 0$, that is, $Fy - y^* = 0$ and $Ky^* + y = 0$. Hence, $y^* = Fy$ and $y + KFy = 0$.

3 Main Results

In theorems 3.1 and 3.2 below, the sequences $\{\alpha_n\}_{n=1}^{\infty}, \{\lambda_n\}_{n=1}^{\infty}$ and $\{\theta_n\}_{n=1}^{\infty}$ are in $(0,1)$ and are assumed to satisfy the following conditions:

(i) $\alpha_n, \lambda_n, \theta_n \to 0$ as $n \to \infty$, $(\frac{\theta_{n-1}-\theta_n}{\theta_n}K) \le 1$, $\frac{\theta_n}{\alpha_n} \to 0$ as $n \to \infty$, $\sum_{n=1}^{\infty} \lambda_n\theta_n = \infty$;

(ii) $\lambda_n \le \gamma_0\theta_n$, $[\delta_E^{-1}(\lambda_n M_1^*) + \delta_{E^*}^{-1}(\lambda_n M_2^*)] \le \gamma_0\theta_n$;

(iii) $\sum_{n=1}^{\infty} \delta_E^{-1}(\lambda_n M_1^*) < \infty$, $\sum_{n=1}^{\infty} \delta_{E^*}^{-1}(\lambda_n M_2^*) < \infty$;

(iv) $\frac{\delta_E^{-1}\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}K\right)}{\lambda_n\theta_n} \to 0$, $\frac{\delta_{E^*}^{-1}\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}K\right)}{\lambda_n\theta_n} \to 0$ as $n \to \infty$,

for some constants $M_1^* > 0$, $M_2^* > 0$, $K > 0$ and $\gamma_0 > 0$; where δ_E is the modulus of convexity of E and δ_{E^*} is the modulus of convexity of E^*.

Theorem 3.1. *Let E be a uniformly smooth and uniformly convex real Banach space and $F : E \to E^*$, $K : E^* \to E$ be maximal monotone and bounded maps. For $u_1 \in E$, $, v_1 \in E^*$, define the sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ in E and E^*, respectively by*

$$u_{n+1} = J^{-1}(Ju_n - \lambda_n\alpha_n(Fu_n - v_n) - \lambda_n\theta_n(Ju_n - Ju_1)),\ n \ge 1,$$

$$v_{n+1} = J_*^{-1}(J_*v_n - \lambda_n\alpha_n(Kv_n + u_n) - \lambda_n\theta_n(J_*v_n - J_*v_1)),\ n \ge 1,$$

Assume that the equation $u + KFu = 0$ has a solution. Then, the sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are bounded.

Proof. For (u_n, v_n), $(u^*, v^*) \in E \times E^*$ where u^* is a solution of (1.8) with $v^* = Fu^*$, set $w_n = (u_n, v_n)$ and $w^* = (u^*, v^*)$. Define $\Lambda : (E \times E^*) \times (E \times E^*) \to \mathbb{R}$ by

$$\Lambda(w_1, w_2) = \phi(u_1, u_2) + \phi(v_1, v_2), \tag{3.1}$$

where $w_1 = (u_1, v_1)$ and $w_2 = (u_2, v_2)$. Let $E \times E^*$ be endowed with the norm $\|(u,v)\| = (\|u\|_E^2 + \|v\|_{E^*}^2)^{\frac{1}{2}}$. We show that $\Lambda(w^*, w_n) \le r$, for all $n \ge 1$ and for some $r > 0$.
Using the fact that F and K are bounded, define

$$M_1 := \sup\{||\alpha(Fu - v) + \theta(Ju - Ju_1)|| : (u,v) \in B_{E\times E^*},\ \alpha, \theta \in (0,1)\} + 1;$$

$$M_2 := \sup\{||\alpha(Kv + u) + \theta(J_*v - J_*v_1)|| : (u,v) \in B_{E\times E^*},\ \alpha, \theta \in (0,1)\} + 1;$$

$$M_3 := \sup\{||(Ju - Ju_1|| : ||u|| \le r_0\} + 1,\ for\ some\ r_0 > 0;$$

$$M_4 := \sup\{||J^{-1}(Ju - \lambda\alpha(Fu - v) - \lambda\theta(Ju - Ju_1)) - u|| : (u,v) \in B_{E\times E^*}, \lambda, \alpha, \theta \in (0,1)\} + 1;$$

$$M_5 := \sup\{||Jv - Jv_1|| : ||v|| \le r_0^*\} + 1,\ for\ some\ r_0^* > 0;$$

$$M_6 := \sup\{||J_*^{-1}(J_*v - \lambda\alpha(Kv + u) - \lambda\theta(J_*v - J_*v_1)) - v|| : (u,v) \in B_{E\times E^*}, \lambda, \alpha, \theta \in (0,1)\} + 1;$$

$$M_1^* = 2LM_1M_4$$

$$M_2^* = 2LM_2M_6$$

$$M^* := \max\{2c_2M_1, 2c_2M_2, 2c_2M_3, 2c_2M_5, 2M_1M_4 + 2M_2M_6\}$$

where c_2 *and* L are constants appearing in Lemma 2.5 and $B_{E\times E^*} = \{w \in E \times E^* : \Lambda(w^*, w) \le r\}$.
Let $r > 0$ be such that

$$\frac{r}{5} \ge \Lambda(w^*, w_1).$$

Define

$$\gamma_0 := \min\Big\{1, \frac{r}{5M^*}, \frac{1}{M_1^*}, \frac{1}{M_2^*}\Big\}.$$

Claim: $\Lambda(w^*, w_n) \leq r, \ \forall\ n \geq 1.$
The proof of this claim is by induction. By construction, we have $\Lambda(w^*, w_1) \leq r$.
Assume that $\Lambda(w^*, w_n) \leq r$ for some $n \geq 1$. This implies that

$$\phi(u^*, u_n) + \phi(v^*, v_n) \leq r, \ \textit{for some}\ n \geq 1.$$

We prove that $\Lambda(w^*, w_{n+1}) \leq r$. Suppose, for contradiction, that this is not the case, then $\Lambda(w^*, w_{n+1}) > r$. From lemma (2.5), we have that

$$\begin{aligned}\delta_E(c_2^{-1}||u_{n+1} - u_n||) \qquad &\leq 2L||Ju_{n+1} - Ju_n||||u_{n+1} - u_n|| \\ &\leq \lambda_n 2LM_1M_4.\end{aligned}$$

This yields

$$||u_{n+1} - u_n|| \leq c_2\delta_E^{-1}(\lambda_n M_1^*). \tag{3.2}$$

Also, using lemma 2.6, we obtain

$$||v_{n+1} - v_n|| \leq c_2\delta_{E^*}^{-1}(\lambda_n M_2^*). \tag{3.3}$$

Using the definition of u_{n+1}, equation (2.4) and inequality (2.5) with

$$y^* = \lambda_n\alpha_n(Fu_n - v_n) + \lambda_n\theta_n(Ju_n - Ju_1),$$

we obtain:

$$\begin{aligned}\phi(u^*, u_{n+1}) \quad &= \quad \phi(u^*, J^{-1}(Ju_n - \lambda_n\alpha_n(Fu_n - v_n) - \lambda_n\theta_n(Ju_n - Ju_1))) \\ &\leq \quad V(u^*, Ju_n) - 2\Big\langle u_{n+1} - u^*, \lambda_n\alpha_n(Fu_n - v_n) + \lambda_n\theta_n(Ju_n - Ju_1)\Big\rangle \\ &= \quad \phi(u^*, u_n) - 2\Big\langle u_{n+1} - u_n, \lambda_n\Big(\alpha_n(Fu_n - v_n) + \theta_n(Ju_n - Ju_1)\Big)\Big\rangle \\ &\quad\ -2\Big\langle u_{n+1} - u^*, \lambda_n\Big(\alpha_n(Fu_n - v_n) + \theta_n(Ju_n - Ju_1)\Big)\Big\rangle \\ &\leq \quad \phi(u^*, u_n) + 2\|u_{n+1} - u_n\|\Big|\Big|\lambda_n\Big(\alpha_n(Fu_n - v_n) + \theta_n(Ju_n - Ju_1\Big)\Big|\Big| \\ &\quad\ -2\lambda_n\Big\langle u_n - u^*, \alpha_n(Fu_n - v_n) + \theta_n(Ju_n - Ju_1)\Big\rangle\end{aligned}$$

Which implies that

$$\begin{aligned}\phi(u^*, u_{n+1}) \quad &\leq \quad \phi(u^*, u_n) + 2\|u_{n+1} - u_n\|\lambda_n M_1 \\ &\quad\ -2\lambda_n\Big\langle u_n - u^*, \alpha_n(Fu_n - v_n) + \theta_n(Ju_n - Ju_1)\Big\rangle.\end{aligned} \tag{3.4}$$

Observe that using the monotonicity of F and J, we have:

$$\begin{aligned}&-2\lambda_n\Big\langle u_n - u^*, \alpha_n(Fu_n - v_n) + \theta_n(Ju_n - Ju_1)\Big\rangle \\ \leq \quad &-2\lambda_n\alpha_n\langle u_n - u^*, (Fu^* - v_n)\rangle - 2\lambda_n\theta_n\langle u_n - u_{n+1}, Ju_n - Ju_{n+1}\rangle \\ &-2\lambda_n\theta_n\langle u_n - u_{n+1}, Ju_{n+1} - Ju_1\rangle - 2\lambda_n\theta_n\langle u_{n+1} - u^*, Ju_n - Ju_{n+1}\rangle \\ &-2\lambda_n\theta_n\langle u_{n+1} - u^*, Ju_{n+1} - Ju_1\rangle \\ \leq \quad &-2\lambda_n\alpha_n\langle u_n - u^*, (Fu^* - v_n)\rangle + 2\lambda_n\theta_n||u_n - u_{n+1}||||Ju_{n+1} - Ju_1|| \\ &+2\lambda_n\theta_n||u_{n+1} - u^*||||Ju_n - Ju_{n+1}|| - 2\lambda_n\theta_n\langle u_{n+1} - u^*, Ju_{n+1} - Ju_1\rangle.\end{aligned}$$

Substituting into inequality (3.4), we obtain

$$\begin{aligned}\phi(u^*, u_{n+1}) \quad \le \quad & \phi(u^*, u_n) + 2\|u_{n+1} - u_n\|\lambda_n M_1 - 2\lambda_n\alpha_n\langle u_n - u^*, (Fu^* - v_n)\rangle \\ & +2\lambda_n\theta_n||u_n - u_{n+1}||||Ju_{n+1} - Ju_1|| + 2\lambda_n\theta_n||u_{n+1} - u^*||||Ju_n - Ju_{n+1}|| \\ & -2\lambda_n\theta_n\langle u_{n+1} - u^*, Ju_{n+1} - Ju_1\rangle.\end{aligned}$$

Now, using inequality (2.10) of lemma 2.7 and inequality (3.2), we have that

$$\begin{aligned}\phi(u^*, u_{n+1}) \quad \le \quad & \phi(u^*, u_n) - \lambda_n\theta_n\phi(u^*, u_{n+1}) + \lambda_n\theta_n\phi(u^*, u_1) \\ & +\lambda_n\delta_E^{-1}(\lambda_n M_1^*)(2c_2M_1) + 2\lambda_n\theta_n(\lambda_n M_1)M_4 \\ & +\lambda_n\theta_n[\delta_E^{-1}(\lambda_n M_1^*)(2c_2M_3)] - 2\lambda_n\alpha_n\langle u_n - u^*, (Fu^* - v_n)\rangle.\end{aligned} \tag{3.5}$$

Similarly, using the fact that K and J_* are monotone, inequality (2.10) of lemma 2.7 and inequality (3.3), we have

$$\begin{aligned}\phi(v^*, v_{n+1}) \quad \le \quad & \phi(v^*, v_n) - \lambda_n\theta_n\phi(v^*, v_{n+1}) + \lambda_n\theta_n\phi(v^*, v_1) \\ & +\lambda_n\delta_{E^*}^{-1}(\lambda_n M_2^*)(2c_2M_2) + 2\lambda_n\theta_n(\lambda_n M_2)M_6 \\ & +\lambda_n\theta_n[\delta_{E^*}^{-1}(\lambda_n M_2^*)(2c_2M_5)] - 2\lambda_n\alpha_n\langle v_n - v^*, (Kv^* + u_n)\rangle.\end{aligned} \tag{3.6}$$

Observe that since $u^* + KFu^* = 0$, setting $Fu^* = v^*$, we obtain that $Kv^* = -u^*$, and these equations yield

$$2\lambda_n\alpha_n\langle u_n - u^*, (v_n - Fu^*)\rangle + 2\lambda_n\alpha_n\langle v_n - v^*, -(Kv^* + u_n)\rangle = 0.$$

Adding (3.5) and (3.6), we obtain

$$\begin{aligned}r \quad < \quad & \Lambda(w^*, w_{n+1}) \\ \le \quad & \Lambda(w^*, w_n) - \lambda_n\theta_n\Lambda(w^*, w_{n+1}) + \lambda_n\theta_n\Lambda(w^*, w_1) + \lambda_n[\delta_E^{-1}(\lambda_n M_1^*) + \delta_{E^*}^{-1}(\lambda_n M_2^*)]M^* \\ & +\lambda_n\theta_n[\delta_E^{-1}(\lambda_n M_1^*) + \delta_{E^*}^{-1}(\lambda_n M_2^*)]M^* + \lambda_n\theta_n(\lambda_n M^*).\end{aligned}$$

So that

$$\begin{aligned}r < \Lambda(w^*, w_{n+1}) \quad \le \quad & \Lambda(w^*, w_n) - \lambda_n\theta_n\Lambda(w^*, w_{n+1}) + \lambda_n\theta_n\Lambda(w^*, w_1) \\ & +\lambda_n\theta_n(\gamma_0\theta_n)M^* + \lambda_n(\theta_n\gamma_0)M^* + \lambda_n\theta_n\gamma_0 M^* \\ \le \quad & r - \lambda_n\theta_n r + \lambda_n\theta_n\frac{r}{5} + \lambda_n\theta_n\frac{r}{5} + \lambda_n\theta_n\frac{r}{5} + \lambda_n\theta_n\frac{r}{5} \\ < \quad & r.\end{aligned}$$

This is a contradiction, hence, $\Lambda(w^*, w_{n+1}) \le r$ and so $\Lambda(w^*, w_n) \le r$ for all $n \ge 1$. As a result, we have $\phi(u^*, u_n) \le r$ and $\phi(v^*, v_n) \le r$ for all $n \ge 1$. Thus from inequality (2.2), we have that $\{u_n\}_{n\ge1}$ and $\{v_n\}_{n\ge1}$ are bounded. □

Theorem 3.2. *Let E be a uniformly convex and uniformly smooth real Banach space and $F : E \to E^*$, $K : E^* \to E$ be maximal monotone and bounded maps. For $u_1 \in E$, $v_1 \in E^*$, define the sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ in E and E^*, respectively by*

$$u_{n+1} = J^{-1}(Ju_n - \lambda_n\alpha_n(Fu_n - v_n) - \lambda_n\theta_n(Ju_n - Ju_1)),\ n \ge 1,$$

$$v_{n+1} = J_*^{-1}(J_*v_n - \lambda_n\alpha_n(Kv_n + u_n) - \lambda_n\theta_n(J_*v_n - J_*v_1)),\ n \ge 1,$$

Assume that the equation $u + KFu = 0$ has a solution. Then, the sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ converge strongly to u^* and v^*, respectively, where u^* is the solution of $u+KFu = 0$ with $v^* = Fu^*$.

Proof. Using equation (2.4), lemmas 2.9 and 2.1, with $y^* = \lambda_n\alpha_n(Fu_n - v_n) + \lambda_n\theta_n(Ju_n - Ju_1)$, we have

$$\begin{aligned}
\phi(y_n, u_{n+1}) &= \phi(y_n, J^{-1}(Ju_n - \lambda_n\alpha_n(Fu_n - v_n) - \lambda_n\theta_n(Ju_n - Ju_1))) \\
&\leq V(y_n, Ju_n) - 2\langle u_{n+1} - y_n, \lambda_n\alpha_n(Fu_n - v_n) + \lambda_n\theta_n(Ju_n - Ju_1)\rangle \\
&= \phi(u_n, y_n) + 2\langle u_n, Jy_n\rangle - 2\langle y_n, Ju_n\rangle - 2\lambda_n\langle u_{n+1} - y_n, \alpha_n(Fu_n - v_n) + \theta_n(Ju_n - Ju_1)\rangle \\
&= V(u_n, Jy_n) + 2\langle u_n, Jy_n\rangle - 2\langle y_n, Ju_n\rangle - 2\lambda_n\langle u_{n+1} - y_n, \alpha_n(Fu_n - v_n) + \theta_n(Ju_n - Ju_1)\rangle \\
&\leq V(u_n, Jy_{n-1}) - 2\langle y_n - u_n, Jy_{n-1} - Jy_n\rangle + 2\langle u_n, Jy_n)\rangle - 2\langle y_n, Ju_n)\rangle \\
&\quad -2\lambda_n\langle u_{n+1} - y_n, \alpha_n(Fu_n - v_n) + \theta_n(Ju_n - Ju_1)\rangle \\
&= \phi(y_{n-1}, u_n) + 2\langle y_{n-1}, Ju_n\rangle - 2\langle u_n, Jy_{n-1}\rangle - 2\langle y_n - u_n, Jy_{n-1} - Jy_n\rangle \\
&\quad +2\langle u_n, Jy_n\rangle - 2\langle y_n, Ju_n\rangle - 2\lambda_n\langle u_{n+1} - y_n, \alpha_n(Fu_n - v_n) + \theta_n(Ju_n - Ju_1)\rangle \\
&= \phi(y_{n-1}, u_n) + 2\langle y_{n-1} - y_n, Ju_n\rangle + 2\langle y_n, Jy_n - Jy_{n-1}\rangle \\
&\quad -2\lambda_n\langle u_{n+1} - y_n, \alpha_n(Fu_n - v_n) + \theta_n(Ju_n - Ju_1)\rangle.
\end{aligned}$$

Appying monotonicity of F and using equations (2.16), (2.11), (3.2), (2.13) and (2.14), we have

$$\begin{aligned}
\phi(y_n, u_{n+1}) &\leq \phi(y_{n-1}, u_n) + ||y_n - y_{n-1}||C_1 + ||Jy_n - Jy_{n-1}||C_2 + 2\lambda_n||u_{n+1} - u_n||M_1 \\
&\quad -2\lambda_n\langle u_n - y_n, \alpha_n(Fu_n - v_n) + \theta_n(Ju_n - Ju_1)\rangle \\
&= \phi(y_{n-1}, u_n) + ||y_n - y_{n-1}||C_1 + ||Jy_n - Jy_{n-1}||C_2 + 2\lambda_n||u_{n+1} - u_n||M_1 \\
&\quad -2\lambda_n\langle u_n - y_n, \alpha_n(Fu_n - v_n) + \theta_n(Ju_n - Jy_n - \frac{\alpha_n}{\theta_n}(Fy_n - y_n^*))\rangle \\
&\leq \phi(y_{n-1}, u_n) + ||y_n - y_{n-1}||C_1 + ||Jy_n - Jy_{n-1}||C_2 + 2\lambda_n||u_{n+1} - u_n||M_1 \\
&\quad -2\lambda_n\alpha_n\langle u_n - y_n, y_n^* - v_n\rangle - 2\lambda_n\theta_n\langle u_n - y_{n-1}, Ju_n - Jy_{n-1}\rangle \\
&\quad -2\lambda_n\theta_n\langle u_n - y_{n-1}, Jy_{n-1} - Jy_n\rangle - 2\lambda_n\theta_n\langle y_{n-1} - y_n, Ju_n - Jy_n\rangle \\
&\leq \phi(y_{n-1}, u_n) + ||y_n - y_{n-1}||C_1 + ||Jy_n - Jy_{n-1}||C_2 + 2\lambda_n||u_{n+1} - u_n||M_1 \\
&\quad -\lambda_n\theta_n\phi(y_{n-1}, u_n) + ||Jy_n - Jy_{n-1}||C_3 + ||y_n - y_{n-1}||C_4 - 2\lambda_n\alpha_n\langle u_n - y_n, y_n^* - v_n\rangle \\
&\leq \phi(y_{n-1}, u_n) - \lambda_n\theta_n\phi(y_{n-1}, u_n) + \delta_E^{-1}\Big(\frac{\theta_{n-1} - \theta_n}{\theta_n}K\Big)C_5 \qquad (3.7) \\
&\quad +\delta_{E^*}^{-1}\Big(\frac{\theta_{n-1} - \theta_n}{\theta_n}K\Big)C_6 + 2c_2\lambda_n\delta_E^{-1}(\lambda_n M_1^*)M_1 - 2\lambda_n\alpha_n\langle u_n - y_n, y_n^* - v_n\rangle,
\end{aligned}$$

where $C_1,\ C_2,\ C_3,\ C_4$ are positive constants and $C_5 = c_2C_1 + c_2C_4,\ C_6 = c_2C_2 + c_2C_3$.

Similarly, applying monotonicity of K and using equations (2.17), (2.11), (3.3), (2.13) and (2.14), we have

$$\begin{aligned}
\phi(y_n^*, v_{n+1}) &\leq \phi(y_{n-1}^*, v_n) - \lambda_n\theta_n\phi(y_{n-1}^*, v_n) + \delta_E^{-1}\Big(\frac{\theta_{n-1} - \theta_n}{\theta_n}K\Big)C_5^* \qquad (3.8) \\
&\quad +\delta_{E^*}^{-1}\Big(\frac{\theta_{n-1} - \theta_n}{\theta_n}K\Big)C_6^* + 2c_2\lambda_n\delta_{E^*}^{-1}(\lambda_n M_2^*)M_2 - 2\lambda_n\alpha_n\langle v_n - y_n^*, u_n - y_n\rangle.
\end{aligned}$$

where C_5^* and C_6^* are positive constants.

Hence, adding eqautions (3.7) and (3.8) we have

$$\begin{aligned}
\Lambda(p_n, w_{n+1}) &\leq \Lambda(p_{n-1}, w_n) - \lambda_n\theta_n\Big(\phi(y_{n-1}, u_n) + \phi(y_{n-1}^*, v_n)\Big) + 2c_2\lambda_n\delta_E^{-1}(\lambda_n M_1^*)M_1 \\
&\quad +2c_2\lambda_n\delta_{E^*}^{-1}(\lambda_n M_2^*)M_2 + \delta_E^{-1}\Big(\frac{\theta_{n-1} - \theta_n}{\theta_n}K\Big)C_5 + \delta_E^{-1}\Big(\frac{\theta_{n-1} - \theta_n}{\theta_n}K\Big)C_5^* \\
&\quad +\delta_{E^*}^{-1}\Big(\frac{\theta_{n-1} - \theta_n}{\theta_n}K\Big)C_6 + \delta_{E^*}^{-1}\Big(\frac{\theta_{n-1} - \theta_n}{\theta_n}K\Big)C_6^*.
\end{aligned}$$

where $p_n = [y_n, y_n^*]$ is as in remark 2.2. Letting $M^* = \max\{C_5 + C_5^*, C_6 + C_6^*, 2c_2M_1, 2c_2M_2\}$, we have

$$\begin{aligned}\Lambda(p_n, w_{n+1}) \quad &\leq \quad \Lambda(p_{n-1}, w_n) - \lambda_n\theta_n\Lambda(p_{n-1}, w_n) + \lambda_n\delta_E^{-1}(\lambda_n M_1^*)M^* + \lambda_n\delta_{E^*}^{-1}(\lambda_n M_2^*)M^* \\ &\quad + \delta_E^{-1}\Big(\frac{\theta_{n-1} - \theta_n}{\theta_n}K\Big)M^* + \delta_{E^*}^{-1}\Big(\frac{\theta_{n-1} - \theta_n}{\theta_n}K\Big)M^* \\ &\leq \quad \Lambda(p_{n-1}, w_n) - \lambda_n\theta_n\Lambda(p_{n-1}, w_n) + \delta_E^{-1}(\lambda_n M_1^*)M^* + \delta_{E^*}^{-1}(\lambda_n M_2^*)M^* \\ &\quad + \delta_E^{-1}\Big(\frac{\theta_{n-1} - \theta_n}{\theta_n}K\Big)M^* + \delta_{E^*}^{-1}\Big(\frac{\theta_{n-1} - \theta_n}{\theta_n}K\Big)M^*.\end{aligned}$$

Setting

$\rho_n := \Lambda(p_{n-1}, w_n)$; $\beta_n := \lambda_n\theta_n$; $\zeta_n := \left(\frac{\delta_E^{-1}\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}K\right)M^*}{\lambda_n\theta_n} + \frac{\delta_{E^*}^{-1}\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}K\right)M^*}{\lambda_n\theta_n}\right)$;
$\gamma_n := \delta_E^{-1}(\lambda_n M_1^*)M^* + \delta_{E^*}^{-1}(\lambda_n M_2^*)M^*$;
we have

$$\rho_{n+1} \leq (1 - \beta_n)\rho_n + \beta_n\zeta_n + \gamma_n,\ n \geq 1.$$

It now follows from Lemma (2.4) that $\rho_n \to 0$ as n$\to \infty$, *i.e.*, $\Lambda(p_{n-1}, w_n) \to 0$ as n$\to \infty$. Consequently, by lemma (2.3), we obtain that $\lim||u_n - y_{n-1}|| = 0$. Hence using remark 2.3, we have that the sequence $\{u_n\}_{n=1}^{\infty}$ converges strongly to a solution of (1.8). □

Remark 3.1. We have (see e.g., Alber [29]) for $p > 1, q > 1, X = L^p, X^* = L^q,$ *that*

$$\delta_{X^*}(\epsilon) = 1 - \Big(1 - \Big(\frac{\epsilon}{2}\Big)^q\Big)^{\frac{1}{q}},$$

and thus obtain also that

$$\delta_{X^*}^{-1}(\epsilon) = 2[1 - (1 - \epsilon)^q]^{\frac{1}{q}} \leq 2q^{\frac{1}{q}}\epsilon^{\frac{1}{q}}.$$

(The last inequality follows since $(1 - \epsilon)^q > 1 - q\epsilon$, for $q > 1$).

Prototypes for our result are the following:

$$\theta_n = \frac{1}{(n+1)^b},\ \lambda_n = \frac{1}{(n+1)^a}\ and\ \alpha_n = \frac{1}{(n+1)^\gamma}\quad n \geq 1,$$

where

$$\gamma > 0,\ \gamma < b < \frac{a}{r},\ a + b < \frac{1}{r},\ b < \frac{1}{K};\ where\ K > 0\ is\ as\ defined\ in\ lemma\ 2.5,\ r = max\{p, q\}.$$

For example, without loss of generality, if we set $r = p$, then takng

$$a := \frac{1}{(p+1)};\quad b := min\Big\{\frac{1}{2K}, \frac{1}{2p(p+1)}\Big\},$$

conditions (i) to (iv) are satisfied.

Remark 3.2. (see e.g., Alber [29], p.36) The analytical representations of duality mappings are known in a number of Banach spaces. For instance, in the spaces l^p, $L^p(G)$ and $W_m^p(G)$, $p \in (1, \infty)$, $p^{-1} + q^{-1} = 1$, respectively,

$$Jx = ||x||_{l^p}^{2-p}y \in l^q,\quad y = (|x_1|^{p-2}x_1, |x_2|^{p-2}x_2, ...),\quad x = (x_1, x_2, ...),$$
$$Jx = ||x||_{L^p}^{2-p}|x(s)|^{p-2}x(s) \in L^q(G),\quad s \in G,$$

and

$$Jx = ||x||_{W_m^p}^{2-p}\sum_{|\alpha|\leq m}(-1)^{|\alpha|}D^\alpha(|D^\alpha x(s)|^{p-2}D^\alpha x(s)) \in W_{-m}^q(G), m > 0, s \in G.$$

4 Conclusion

Theorem 3.2 complements Theorem 1.3 to provide iterative methods for the approximation of solutions of the Hammerstein equation $u + KFu = 0$ in more general spaces when the operators K and F are *bounded maximal monotone-type operators.*

Acknowledgement

The authors wish to thank the referees for their comments and suggestions.

Competing Interests

The authors have declared that no competing interests exist.

References

[1] Zarantonello EH. Solving functional equations by contractive averaging. Tech. Rep. 160, U. S. Army Math. Research Center, Madison, Wisconsin; 1960.

[2] Minty GJ. Monotone (nonlinear) operators in Hilbert space. Duke Math. J. 1962;29(4):341-346.

[3] Kačurovskii RI. On monotone operators and convex functionals. Uspekhi Mathematicheskikh Nauk. 1960;15(4):213-215.

[4] Pascali D, Sburian S. Nonlinear mappings of monotone type. Editura Academia Bucuresti, Romania; 1978.

[5] Chidume CE, Djitte N. Iterative approximation of solutions of nonlinear equations of Hammerstein-type. Nonlinear Anal. 2009;70:4086-4092.

[6] Chidume CE, Djitte N. Approximation of solutions of Hammerstein equations with bounded strongly accretive nonlinear operator. Nonlinear Anal. 2009;70:4071-4078.

[7] Chidume CE, Shehu Y. Approximation of solutions of generalised equations of Hammerstein-type. Comput. Math. Appl. 2012;63:966-974.

[8] Dolezale V. Monotone operators and its applications in automation and network theory. Studies in Automation and Control, Elsevier Science Publ., New York. 1979;3.

[9] Brézis H, Browder FE. Some new results about Hammerstein Equations. Bull. Amer. Math. Soc. 1974;80:567-572.

[10] Brézis H, Browder FE. Existence theorems for nonlinear integral equations of Hammerstein type. Bull. Amer. Math. Soc. 1975;81:73-78.

[11] Brézis H, Browder FE. Nonlinear integral equations and systems of Hammerstein type. Bull. Amer. Math. Soc. 1976;82:115-147.

[12] Browder FE. Nonlinera functional analysis and nonlinear integral equations of Hammerstein and Uryshon type. Contributions to Nonlinear Functional Analysis, Academic Press. 1971;425-500.

[13] Browder FE, De Figueiredo DG, Gupta CP. Maximal mono-tone operators and nonlinear integral equations of Hammerstein type. Bull. Amer. Math. Soc. 1970;76:700-705.

[14] Mann WR. Mean value methods in iteration. Proc. Amer. Math. Soc. 1953;4:506-510.

[15] Chidume CE, Osilike MO. Iterative solutions of nonlinear integral equations of Hammerstein-type. J. Nigerian Math. Soc. 1992;11:9-18 (MR96c:65207).

[16] Chidume CE, Zegeye H. Iterative approximation of solutions of nonlinear equation of Hammerstein-type. Abstr. Appl. Anal. 2003;6:353-367.

[17] Chidume CE, Zegeye H. Approximation of solutions of nonlinear equations of monotone and Hammerstein-type. Appl. Anal. 2003;82(8):747-758.

[18] Chidume CE, Zegeye H. Approximation of solutions of nonlinear equations of Hammerstein-type in Hilbert space. Proc. Amer. Math. Soc. 2005;133(3):851-858.

[19] Chidume CE, Ofeodu EU. Solution of nonlinear integral equations of Hammerstein-type. Nonlinear Anal. 2011;74:4293-4299.

[20] Chidume CE, Osilike MO. Iterative solutions of nonlinear accretive operator equations in arbitrary Banach spaces. Nonlinear Analysis-Theory Methods & Applications. 1999;36:863-872.

[21] Chidume CE. Geometric properties of banach spaces and nonlinear iterations. Lectures Notes in Mathematics, Springer, London, UK. 2009;1965.

[22] Lindenstrauss J, Tzafriri L. Classical Banach spaces II: Function Spaces. Ergebnisse Math. Grenzgebiete Bd. 97, Springer-Verlag, Berlin; 1979.

[23] Ya Alber. Metric and generalized projection operators in Banach spaces: Properties and applications. In Theory and Applications of Nonlinear Operators of Accretive and Monotone Type (A. G. Kartsatos, Ed.), Marcel Dekker, New York. 1996;15-50.

[24] Ya Alber, Guerre-Delabriere S. On the projection methods for fixed point problems. Analysis (Munich). 2001;21(1):17-39.

[25] Kamimura S, Takahashi W. Strong convergence of a proximal-type algorithm in a Banach space. SIAMJ. Optim. 2002;13(3):938-945.

[26] Reich S. A weak convergence theorem for the alternating methods with Bergman distance. in: A. G. Kartsatos (Ed.), Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, in Lecture notes in pure and Appl. Math., Dekker, New York. 1996;178:313-318.

[27] Xu HK. Iterative algorithms for nonlinear operators. J. London Math. Soc. 2002;66(2):240-256.

[28] Reich S. Constructive techniques for accretive and monotone operators. Applied Non-linear Analysis, Academic Press, New York. 1979;335-345.

[29] Ya Alber, Ryazantseva I. Nonlinear Ill Posed Problems of Monotone Type. Springer, London, UK; 2006. J. Nonlinear Convex Anal. 2014;15(5):851-865.

[30] Chidume CE, Uba MO, Uzochukwu MI, Otubo EE, Idu KO. A Strong convergence theorem for zeros of maximal monotone maps with applications to convex minimization and variational inequality problems. Proceeding of Edinburgh Mathematical Society (Submitted for publication).

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://sciencedomain.org/review-history/16555