British Journal of Mathematics & Computer Science
19(2): 1-9, 2016; Article no.BJMCS.29322
ISSN: 2231-0851

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Some New Three Step Iterative Methods for Solving Nonlinear Equation Using Steffensen's and Halley Method

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

Article Information

DOI: 10.9734/BJMCS/2016/29322
Editor(s):
(1) Andrej V. Plotnikov, Department of Applied and Calculus Mathematics and CAD, Odessa State Academy of Civil Engineering and Architecture, Ukraine.
(2) Zuomao Yan, Department of Mathematics, Hexi University. China.
Reviewers:
(1) Nasr Al Din IDE, University of Aleppo, Syria.
(2) Fayyaz Ahmad, Politecnical University Catalonia, Barcelona, Spain.
Complete Peer review History: http://www.sciencedomain.org/review-history/16602

Received: 3rd September 2016
Accepted: 29th September 2016
Published: 18th October 2016

Original Research Article

Abstract

In this paper, we introduce the comparative study of new three step iterative methods for finding the zeros of the nonlinear equation $f(x) = 0$. The new method based on the Steffensen's method and Halley method with using predictor – corrector technique. It is established that the new method (NTSM-1) has convergence order sixth and second new method (NTSM-2) has convergence order seventh. Numerical tests show that the new methods is comparable with the well known existing methods and gives better results.

Keywords: *Non linear equations; iterative methods; three step; convergence analysis; Halley method; Steffensen's method.*

1 Introduction

Numerical analysis is the area of mathematics and computer sciences that creates, analyzes and implements algorithms for solving numerically the problems of continuous mathematics. Such problems originate generally from real – world applications of algebra, geometry and calculus and they involve variables which vary continuously: These problems occur throughout the natural sciences, social sciences, engineering, medicine and business. New three step iterative methods for finding the approximate solutions of the

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nonlinear equation $f(x) = 0$ are being developed using several different techniques including Taylor series, quadrature formulas, homotopy and decomposition techniques, see [1-15] and the references therein. The most famous of these methods is the classical Newton's method (NM) [16].

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (1)$$

The Newton's method (1) was modified by Steffensen's method who replaced the first derivative $f'(x)$ in Newton's method by forward difference approximation [16].

$$f'(x) = \frac{f(x_n + f(x_n)) - f(x_n)}{f(x_n)}$$

and obtained the famous Steffensen's method (SM) [17,8,16].

$$x_{n+1} = x_n - \frac{[f(x_n)]^2}{f(x_n + f(x_n)) - f(x_n)} \quad (2)$$

Newton's method and Steffensen's method are of second order converges.

For a given x_0, compute approximation solution x_{n+1} by the iterative scheme

$$x_{n+1} = x_n - \frac{2f(x_n)\, f'(x_n)}{2f'^2(x_n) - f(x_n) f''(x_n)} \quad (3)$$

This is known as Halley's method has cubic convergence (see [7,12]).

We use Predictor – corrector methods, we shall now discuss the application of the explicit and implicit multistep methods, for the solution of the initial value problems. We use explicit (predictor) method for predicting a value and then use the implicit (corrector) method iteratively until the convergence is obtained [20].

2 Iterative Methods

It is well known that a wide class of problems, which arise in various fields of pure and applied sciences can be formulated in terms of nonlinear equations of the type.

$$f(x) = 0 \quad (4)$$

Various numerical methods have been developed using the Taylor series and other techniques. In this paper, we use another series of the nonlinear function $f(x)$ which can be obtained by using the trapezoidal rule and the Fundamental Theorem of Calculus. To be more precise, we assume that α is a simple root of (4) and γ is an initial guess sufficiently close to α now using the trapezoidal rule and fundamental theorem of calculus, one can show that the function $f(x)$ can be approximated by the series [19].

$$f(x) = f(\gamma) + \frac{x-\gamma}{2}\,[f'(x) + f'(\gamma)] \quad (5)$$

where $f'(x)$ is the differential of f .

From (4) and (5), we have

$$x = \gamma - 2\frac{f(\gamma)}{f'(\gamma)} - (x-\gamma)\frac{f'(x)}{f'(\gamma)} \quad (6)$$

Using (6), one can suggest the following iterative method for solving the nonlinear equations (4).

For a given initial choice x_0, find the approximate solution x_{n+1}, by the iterative scheme [19].

$$x_{n+1} = x_n - 2\frac{f(x_n)}{f'(x_n)} - (x_{n+1} - x_n)\ \frac{f'(x_{n+1})}{f'(x_n)}$$

$$n = 0, 1, 2, 3, \ldots\ldots\ldots\ldots\ldots \qquad (7)$$

we use the predictor – corrector technique. Using the Steffensen's method as a predictor, Halley method and equation (6) as a corrector, we suggest and analyze the following iterative method for solving the nonlinear equation (4) and this is the main motivation of this note [17,18,14].

Theorem 1: For a given initial choice x_0, find the approximate solution x_{n+1} by the iterative schemes. From equation (2), (3) and (6).

$$a_n = x_n - \frac{[f(x_n)]^2}{f(x_n + f(x_n)) - f(x_n)}$$

$$b_n = a_n - \frac{2f(a_n)\ f'(a_n)}{2f'^2(a_n) - f(a_n) f''(a_n)}$$

$$x_{n+1} = x_n - 2\frac{f(x_n)}{f'(x_n)} - (b_n - x_n)\ \frac{f'(b_n)}{f'(x_n)}$$

n=0, 1, 2, 3, …………………

Theorem 1 is called the NEW THREE STEP METHOD -1 (NTSM-1) and has sixth order convergence.

Theorem 2: From equation (4) and equation (5) we can have

$$x_{n+1} = x_n - \frac{2\ f(x_n)}{[f'(x_{n+1}) + f'(x_n)]}$$

This is fixed point formulation enable us to suggest the following iterative method for solution the nonlinear equation.

Theorem 3: For a given initial choice x_0, find the approximate solution x_{n+1} by the iterative schemes. From equation (2), (3) and theorem (2)

$$a_n = x_n - \frac{[f(x_n)]^2}{f(x_n + f(x_n)) - f(x_n)}$$

$$b_n = a_n - \frac{2f(a_n)\ f'(a_n)}{2f'^2(a_n) - f(a_n) f''(a_n)}$$

$$x_{n+1} = x_n - \frac{2\ f(x_n)}{[f'(b_n) + f'(x_n)]}$$

n=0, 1, 2, 3, …………………

Theorem 3 is called the NEW THREE STEP METHOD -2 (NTSM-2) and has seventh order convergence.

3 Convergence Analysis

Let us now discuss the convergence analysis of the above Theorem 1.

Theorem 3.1: let $\alpha \epsilon I$ be a simple zero of sufficiently differential function $f : I \subseteq R \rightarrow R$ for an open interval I, if x_0 is sufficiently close to α then the three step iterative method defined by theorem 1 sixth order convergence.

Proof: Let α be a simple zero of f. Than by expanding $f(x_n)$ and $f'(x_n)$ about α we have

$$f(x_n) = e_n c_1 + e_n^2 c_2 + e_n^3 c_3 + \ldots\ldots \quad (8)$$

$$f'(x_n) = 1 + \frac{2c_2}{c_1} e_n + \frac{3c_3}{c_1} e_n^2 + \frac{4c_4}{c_1} e_n^3 + \ldots\ldots \quad (9)$$

Where $c_k = \frac{1}{k!} f^{(k)}(\alpha)$ k=1, 2, 3, …………

and $e_n = x_n - \alpha$

from (8), we have

$$[f(x_n)]^2 = c_1^2 e_n^2 + 2\, c_1 c_2 e_n^3 + c_2^2 e_n^4 + \ldots\ldots \quad (10)$$

$$f(x_n + f(x_n)) = c_1^2 e_n + (3c_1c_2 + c_1^2 c_2 + 2c_2^2) e_n^2 + \ldots\ldots \quad (11)$$

From (10) and (11), we have

$$\frac{[f(x_n)]^2}{f(x_n + f(x_n))} = e_n - \left(\frac{c_2}{c_1} + c_2 + 2\frac{c_2^2}{c_1^2}\right) e_n^2 + \ldots\ldots \quad (12)$$

From (12), we have

$$a_n = \alpha + \left(\frac{c_2}{c_1} + c_2 + 2\frac{c_2^2}{c_1^2}\right) e_n^2 + \ldots\ldots \quad (13)$$

Let us set A = $a_n - \alpha$. Then the equation (13) can be re – written in the form

$$A = \left(\frac{c_2}{c_1} + c_2 + 2\frac{c_2^2}{c_1^2}\right) e_n^2 + \ldots\ldots \quad (14)$$

Now expanding $f(a_n)$, $f'(a_n)$, $f''(a_n)$ about α and using (13), we have

$$f(a_n) = Ac_1 + A^2 c_2 + A^3 c_3 + \ldots\ldots \quad (15)$$

$$f'(a_n) = c_1 + A2c_2 + A^2 3c_3 + A^3 4c_4 + \ldots\ldots \quad (16)$$

$$f''(a_n) = 2c_2 + A6c_3 + A^2 12c_4 + \ldots\ldots \quad (17)$$

Combining (13) – (17), we have

$$b_n = \alpha + (c_2^2 - c_3) A^3 \quad (18)$$

Also expanding $f\ '(b_n)$ about α and using (18), we have

$$f\ '(b_n) = c_1\left[1+2(c_2^2-\ c_3)A^3\ \frac{c_2}{c_1}+\ 3(c_2^2-\ c_3)^2\ A^6\ \frac{c_3}{c_1}+\cdots\right] \tag{19}$$

By substituting (8), (9), (18), (19) in theorem (1) and after some simple calculations, we obtain

$$x_{n+1} = \alpha + (c_2^2-\ c_3)\ (\frac{c_2}{c_1}+c_2+2\frac{c_2^2}{c_1^2})\ e_n^6 + \ \ldots\ldots\ldots \tag{20}$$

$$e_{n+1} = \ (c_2^2-\ c_3)(\frac{c_2}{c_1}+c_2+2\frac{c_2^2}{c_1^2})\ e_n^6 + \ldots\ldots\ldots\ldots \tag{21}$$

This shows that Theorem 1 is sixth order convergence.

Let us now discuss the convergence analysis of the above theorem 3.

Theorem 3.2: Let $\alpha \epsilon I$ be a simple zero of sufficiently differential function $f : I \subseteq R \rightarrow R$ for an open interval I , if x_0 is sufficiently close to α then the three step iterative method defined by theorem 3 seventh order convergence.

Proof:

From equation (19), we have

$$f\ '(b_n) = 2c_1\ e_n + 2\ c_2\ e_n^2 + 2\ c_3\ e_n^3 + 2\ c_4\ e_n^4 + \ldots\ldots.. \tag{22}$$

By substituting (8), (9), (22) in theorem (3) and after some simple calculations, we obtain

$$x_{n+1} = \alpha + 2(c_2^2-\ c_3)\ (\frac{c_2}{c_1}+c_2+2\frac{c_2^2}{c_1^2})\ e_n^7 + \ldots\ldots\ldots\ldots\ldots \tag{23}$$

$$e_{n+1} = 2(c_2^2-\ c_3)\ (\frac{c_2}{c_1}+c_2+2\frac{c_2^2}{c_1^2})\ e_n^7 + \ldots\ldots\ldots\ldots\ldots \tag{24}$$

This shows that Theorem 3 is seventh order convergence.

4 Numerical Examples

For comparisons, we have used Steffensen's method [20] and three step predictor-corrector Newton-Halley method (PCNH) [8] defined respectively by

$$x_{n+1} = x_n - \frac{[f(x_n)]^2}{f(x_n+f(x_n))-f(x_n)}$$

and

$$w_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$y_n\ =\ \ w_n\ -\ \frac{2f(w_n)\ f'(w_n)}{2f'^2(w_n)-f\ (w_n)f''(w_n)},$$

$$x_{n+1} = y_n - 2\,\frac{f(y_n)}{f'(y_n)}\ -\frac{f^2(y_n)f''(y_n)}{2f'(y_n)^3},$$

n = 0, 1, 2,…..

All computations are performed using MATLAB. The following examples are used for numerical testing.

$f_1(x) = 3x - \sqrt{1+\sin(x)}$

$f_2(x) = x + \sin(x) - x^3$

$f_3(x) = e^x - 3x$

$f_4(x) = \cos(x) - 3x + 1$

$f_5(x) = xe^x - 2$

$f_6(x) = xlog_{10}x - 1.2$

$f_7(x) = x - \cos(x)$

$f_8(x) = 4x - e^x$

$f_9(x) = \sin(x) - 1 - x^3$

$f_{10}(x) = e^x - 1.5 - tan^{-1}(x)$

As for the convergence criteria, it was required that the distance of two consecutive approximations δ and also displayed is the number of iterations to approximate the zero (IT), the approximate zero x_n and the value $f(x_n)$.

Table 1. Numerical examples and comparison

Method	IT	x_n	$f(x_n)$	δ
$f_1, x_0 = 0.5$				
SM	4	0.391846907002648	-4.44089209850063e-016	1.01410879693731e-010
PCNH	3	0.391846907002648		2.50388310085725e-008
NTSM-1	2	0.391846907002648		7.53028660899213e-006
NTSM-2	2	0.391846907002648		3.78833973964160e-006
$f_2, x_0 = 1$				
SM	5	1.31716296100603	9.76996261670138e-015	3.32861537577501e-005
PCNH	6	1.31716296100603		4.68398653197255e-011
NTSM-1	3	1.31716296100603		5.34990427110637e-012
NTSM-2	3	1.31716296100603		6.47483999749454e-009
$f_3, x_0 = 0.2$				
SM	4	0.619061286735945	2.22044604925031e-016	7.0389180040209e-009
PCNH	5	0.619061286735945		6.835598753691e-011
NTSM-1	3	0.619061286735945		3.10652923940502e-007
NTSM-2	3	0.619061286735945		3.59104859182224e-008
$f_4, x_0 = 0.2$				
SM	5	0.607101648103123	-1.33226762955019e-015	9.99200722162641e-016
PCNH	4	0.607101648103123		5.16754083967896e-010
NTSM-1	3	0.607101648103123		6.95768997971413e-011
NTSM-2	3	0.607101648103123		3.65396601864632e-012

Method	IT	x_n	$f(x_n)$	δ
$f_5, x_0 = 1$				
SM	6	0.852605502013726	-2.44249065417534e-015	3.517610647279e-010
PCNH	4	0.852605502013726		2.43496630636386e-008
NTSM-1	4	0.852605502013726		2.68309041651094e-010
NTSM-2	3	0.852605502013726		2.88919999036352e-011
$f_6, x_0 = 2$				
SM	4	2.74064609597369	-3.10862446895044e-015	1.70161289503313e-008
PCNH	4	2.74064609597369		1.95032203720302e-008
NTSM-1	3	2.74064609597369		3.98418986691024e-009
NTSM-2	3	2.74064609597369		1.98129956885396e-010
$f_7, x_0 = 0.5$				
SM	5	0.739085133215161	6.66133814775094e-016	7.13984427136438e-013
PCNH	4	0.739085133215161		7.05561942204724e-010
NTSM-1	3	0.739085133215161		4.4899972628798e-011
NTSM-2	3	0.739085133215161		2.45103937146496e-012
$f_8, x_0 = 1.8$				
SM	6	2.15329236411035	-1.77635683940025e-015	5.27400345617934e-011
PCNH		Divergent		
NTSM-1	3	2.15329236411035		1.48246099840321e-008
NTSM-2	3	2.15329236411035		2.29886798486234e-008
$f_9, x_0 = -1$				
SM	4	-1.24905214850119	-1.99840144432528e-014	0.000227912085001725
PCNH	5	-1.24905214850119		2.53839993469285e-009
NTSM-1	3	-1.24905214850119		1.20446399520802e-008
NTSM-2	3	-1.24905214850119		1.01799013663140e-009
$f_{10}, x_0 = 1$				
SM	5	0.767653266201279	0	2.27928786955545e-012
PCNH	5	0.767653266201279		1.48392409471398e-012
NTSM-1	3	0.767653266201279		9.48948919443637e-010
NTSM-2	3	0.767653266201279		1.48009937639415e-010

5 Conclusion

In this paper, we have suggested and analyzed newly developed technique is faster than Steffensen's method (SM) and predictor-corrector Newton-Halley method (PCNH). This method based on a Steffensen's method and Halley method and using predictor – corrector technique. Our method can be considered as significant improvement of Steffensen's method and PCNH and can be considered as alternative method of solving nonlinear equations.

Competing Interests

Authors have declared that no competing interests exist.

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ISSN: 1314-3395 (on-line version)

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(Article in press)

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DOI: 10.1007/s12190-013-0647-3

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The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
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