British Journal of Mathematics & Computer Science
19(2): 1-29, 2016; Article no.BJMCS.29231
ISSN: 2231-0851

SCIENCEDOMAIN *international*
www.sciencedomain.org

QE-Bayesian and E-Bayesian Estimation of the Frechet Model

Hesham M. Reyad[1,2*], Adil M. Younis[1,2] and Soha O. Ahmed[3]

[1]*College of Business and Economics, Qassim University, Kingdom of Saudi Arabia.*
[2]*Faculty of Science, Sudan University of Science and Technology, Sudan.*
[3]*Institute of Statistical Studies and Research, Cairo University, Egypt.*

Authors' contributions

This work was carried out in collaboration among all authors. Author HMR introduced the idea in a methodically structure, did the data analysis and drafted the manuscript. Author AMY assisted in building the study design and also did the final proofreading. Author SOA managed the analyses of the study and literature searches and also proofread the draft. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/BJMCS/2016/29231
Editor(s):
(1) Balswaroop Bhatt, Department of Mathematics and Statistics, The University of the West Indies, St. Augustine, Trinidad (W.I.).
Reviewers:
(1) S. Kim, Wayne State University, Detroit, MI, USA.
(2) Rashmi Awad, Devi Ahilya University, Indore, India.
(3) Hui Li, Icahn Medicine School at Mount Sinai, USA.
Complete Peer review History: http://www.sciencedomains.org/review-history/16611

Original Research Article

Received: 30th August 2016
Accepted: 10th October 2016
Published: 20th October 2016

Abstract

This paper proposes a new technique namely QE-Bayesian estimation, which is a new modification to the E-Bayesian method of estimation. The suggested approach based on replacing the quasi-likelihood function instead of the likelihood function in the E-Bayesian technique. This study is concerned with evaluating the performance of the QE-Bayesian method versus the original E-Bayesian approach in estimating the scale parameter of the Frechet distribution. The QE-Bayes and E-Bayes estimates are obtained under symmetric loss function [squared error loss (SELF)] and three different asymmetric loss functions [entropy loss function (ELF), weighted balanced loss function (WBLF) and minimum expected loss function (MELF)]. The properties of the QE-Bayesian and E-Bayesian estimates are also studied. Comparisons among all estimators are performed in terms of absolute bias(ABias) and mean square error (MSE) via Monte Carlo simulation. Numerical results show that the QE-Bayes estimates are more efficient as compared with the E-Bayes estimates.

Keywords: E-Bayesian estimates; Frechet distribution; loss functions; Monte Carlo simulation; QE-Bayes estimates.

**Corresponding author: E-mail: hesham_reyad@yahoo.com;*

1 Introduction

The Frechet distribution has many applications in modeling extreme events such as, wind speeds, stock exchange and flood, etc. It was first introduced by Maurice Frechet [1]. Several authors studied the Frechet distribution; for examples, Mubarak [2] obtained the maximum likelihood and least squares estimates for Frechet distribution based on progressive type-II censoring. Abbas and Yincai [3] compared the maximum likelihood, probability weighted moments and Bayes estimates for the scale parameter of Frechet distribution in complete samples. Nasir and Aslam [4] estimated the shape parameter of Frechet distribution via Bayesian scheme based on different prior distribution and various loss functions. The probability density function (pdf) of the Frechet distribution is:

$$f(x;\alpha,\beta)=\alpha\beta^{\alpha}\left(\frac{1}{x}\right)^{\alpha+1}e^{-\left(\frac{\beta}{x}\right)^{\alpha}}, \qquad x>0,\ \alpha,\beta>0, \tag{1}$$

Where β and α are the scale and shape parameters respectively.

The quasi-likelihood function was proposed by Wedderbum [5] as a new extension of the maximum likelihood estimation. This method requires a predetermined assumption about mean and variance of the underlying distribution. Given an observation x with mean μ and variance $V(\mu)$, the quasi-likelihood function can be defined as follows:

$$\frac{\partial Q(x;\mu)}{\partial\mu}=\frac{x-\mu}{V(\mu)} \tag{2}$$

equivalent by

$$Q(x;\mu)=\int_{\mu}\frac{x-\mu}{V(\mu)}d\mu+function\ of\ \ X \tag{3}$$

Where $\mu=E(x)$, $V(\mu)=V(x)$. The variance assumption is generalized to $V(x)$ equals $\psi V(\mu)$, where the variance function $V(.)$ is assumed to be known and the parameter ψ may be known or unknown. The quasi-Bayesian estimation approach uses the natural exponential of quasi-likelihood function instead of the ordinary likelihood function.

The E-Bayesian is a new criteria of estimation that was first introduced by Han [6]. This method consists of obtaining the expectation of Bayes estimates with respect to the distributions of hyper parameters. Many authors applied the E-Bayes technique such as, Han [7] estimated the reliability parameter of the exponential distribution by using the E-Bayes and hierarchical Bayes methods based on type-I censored and by considering the quadratic loss function. Yin and Liu [8] applied the E-Bayesian and hierarchical Bayesian estimation techniques to estimate the unknown reliability parameter of geometric distribution based on scaled squared loss function in complete samples. Wei et al. [9] obtained the minimum risk equivariant and E-Bayes estimates for the Burr-XII distribution under entropy loss function in complete samples. Jaheen and Okasha [10] estimated the parameter and reliability function of Burr-XII model via Bayes and E-Bayes techniques based on squared error and LINEX loss functions on type-II censoring. Cai et al. [11] used the E-Bayesian criteria for forecasting of security investment. Okasha [12] obtained the maximum likelihood, Bayesian and E-Bayesian estimates for the parameter, reliability and hazard functions corresponding to the Weibull distribution based on type-II censored data. Azimi et al. [13] derived the Bayes and E-Bayes estimates for parameter and reliability function of the generalized half Logistic model based on progressively type-II censoring and by using symmetric and asymmetric loss functions. Javadkani et al. [14]

constructed the Bayes, empirical Bayes and E-Bayes schemes for estimating shape parameter and reliability function of the two parameter bathtub-shaped lifetime distribution under progressively first-failure-censoring and by using the minimum expected and LINEX loss functions. Liu et al. [15] obtained the E-Bayes and hierarchical estimates for the Rayleigh distribution based on q-symmetric entropy loss function in complete samples. Reyad and Othman [16] derived the Bayesian and E-Bayesian estimates for the shape parameter of the Gumbell type-II distribution under type-II censoring and by considering squared error, LINEX, Degroot, quadratic and minimum expected loss functions. Reyad and Othman [17] studied the Bayes and E-Bayes estimators for the Kumaraswamy distribution under type-II censored data and by using different symmetric and asymmetric loss functions. Reyad et al. [18] compared the Bayes, E-bayes, hierarchical Bayes and empirical Bayes estimates of shape parameter and hazard function associated to the Gompertz model under type-II censoring and by using squared error, quadratic entropy and LINEX loss functions.

The main objective of this paper is to compare the QE-Bayes and E-Bayes methods for estimating the scale parameter corresponding to the Frechet distribution. All estimates are obtained based on symmetric and different asymmetric loss functions. The properties of the QE-Bayes and E-Bayes estimates are investigated.

The remaining of the paper is organized as follow. In Section 2, the different loss functions that will be used in this study are reviewed, the quasi-posterior and posterior distributions are obtained. The QE-Bayesian estimates for β are derived under SELF, WBLF, ELF and MELF in Section 3. In Section 4, the E-Bayesian estimates for β are obtained under SELF, WBLF, ELF and MELF. In Section 5, the properties of the QE-Bayes and E-Bayes estimates are investigated. In Section 6, a Monte Carlo simulation is used to assess the performance of the resulting estimates. Finally, some concluding remarks are presented in Section 7.

2 The Loss Functions, Quasi-posterior and Posterior Distributions

This section contains the different loss functions that will be used in this study, and derivation of the quasi-posterior and posterior distributions corresponding to the Frechet distribution.

2.1 The loss functions

We will use the following loss functions:

2.1.1 The squared error loss function (SELF)

The squared error loss function (SELF) can be defined as follows:

$$L_1(\beta,\hat{\beta}) = (\hat{\beta}-\beta)^2 \tag{4}$$

Where $\hat{\beta}$ is an estimator of β . The Bayes estimator of β relative to the SELF, denoted by $\hat{\beta}_{BS}$, can be obtained by:

$$\hat{\beta}_{BS} = E(\beta|\underline{x}) \tag{5}$$

Provided that the expectation $E(\beta|\underline{x})$ exists and finite.

2.1.2 The weighted balanced loss function (WBLF)

The weighted balanced loss function (WBLF) can be used as Nasir and Aslam [4] to be

$$L_2(\beta,\hat{\beta}) = \left(\frac{\beta-\hat{\beta}}{\hat{\beta}}\right)^2 \tag{6}$$

Where $\hat{\beta}$ is an estimator of β .The Bayes estimator of β relative to WBLF, denoted by $\hat{\beta}_{BW}$, can be obtained by:

$$\hat{\beta}_{BW} = \frac{E(\beta^2|\underline{x})}{E(\beta|\underline{x})} \tag{7}$$

Provided that the expectations $E(\beta^2|\underline{x}), E(\beta|\underline{x})$ exists and finite.

2.1.3 The entropy loss function (ELF)

Day et al. [19] discussed the entropy loss function (ELF) of the form:

$$L_3(\beta,\hat{\beta}) \propto \left(\frac{\hat{\beta}}{\beta}\right) - \ln\left(\frac{\hat{\beta}}{\beta}\right) - 1 \tag{8}$$

Where $\hat{\beta}$ is an estimator of β . The Bayes estimator of β relative to ELF, denoted by $\hat{\beta}_{BE}$, can be obtained by:

$$\hat{\beta}_{BE} = \left[E(\beta^{-1}|\underline{x})\right]^{-1} \tag{9}$$

Provided that the expectation $E(\beta^{-1}|\underline{x})$ exists and finite.

2.1.4 The minimum expected loss function (MELF)

Tummala and Sathe [20] defined the minimum expected loss function (MELF) as follows:

$$L_4(\beta,\hat{\beta}) = \frac{(\hat{\beta}-\beta)^2}{\beta^2} \tag{10}$$

Where $\hat{\beta}$ is an estimator of β . The Bayes estimator of β based on MELF, denoted by $\hat{\beta}_{BM}$, can be obtained by:

$$\hat{\beta}_{BM} = \frac{E(\beta^{-1}|\underline{x})}{E(\beta^{-2}|\underline{x})} \tag{11}$$

Provided that the expectations $E(\beta^{-1}|\underline{x}), E(\beta^{-2}|\underline{x})$ exists and finite.

2.2 The Quasi-posterior distribution

The mean and variance of the Frechet distribution given in (1) are given by:

$$\mu = E(x) = \beta\Gamma(1-\frac{1}{\alpha}), \qquad V(x) = \beta^2\Gamma(1-\frac{2}{\alpha}) - \left[\beta\Gamma(1-\frac{1}{\alpha})\right]^2 = \psi V(\mu) \tag{12}$$

Where

$$\psi = \left[\frac{\Gamma(1-\frac{2}{\alpha})}{\Gamma^2(1-\frac{1}{\alpha})} - 1\right], \qquad V(\mu) = \mu^2 \tag{13}$$

Thus, for a random sample of size n, the quasi-likelihood function can be obtained for the Frechet distribution by using (2) and (3) in (1):

$$\frac{\partial Q(x;\mu)}{\partial \mu} = \frac{\sum_{i=1}^{n} x_i - n\mu}{\mu^2} \tag{14}$$

Which gives

$$Q(x;\mu) = \frac{-\sum_{i=1}^{n} x_i}{\mu} - n\ln\mu \tag{15}$$

We can obtain the quasi-likelihood function in terms of α and β by using (12) in (15) to be

$$Q(x;\alpha,\beta) = \ln\left[\beta\Gamma(1-\frac{1}{\alpha})\right]^{-n} - \frac{\sum_{i=1}^{n} x_i}{\beta\Gamma(1-\frac{1}{\alpha})} \tag{16}$$

The natural exponential of the quasi-likelihood function given in (16) is obtained as follows:

$$\exp\left[Q(x;\alpha,\beta)\right] = \beta^{-n}\Gamma^{-n}(1-\frac{1}{\alpha})\ \exp\left[\frac{-\sum_{i=1}^{n} x_i}{\beta\Gamma(1-\frac{1}{\alpha})}\right] \tag{17}$$

Assuming α is known, then the quasi-likelihood function becomes

$$\exp\left[Q(x;\beta)\right] \propto \beta^{-n} e^{\frac{-D}{\beta}} \tag{18}$$

Where

$$D = \frac{\sum_{i=1}^{n} x_i}{\Gamma(1-\frac{1}{\alpha})} \tag{19}$$

We can use the power density function as a prior distribution of β with rate parameter a and its pdf given by:

$$g(\beta|a) = a\ \beta^{a-1}, \qquad \beta > 0,\ a > 0 \tag{20}$$

The quasi-posterior distribution of β can be obtained by combining (18) and (20):

$$h_1(\beta|\underline{x}) = \frac{\exp[Q(x;\beta)]\, g(\beta|a)}{\int_0^\infty \exp[Q(x;\beta)]\, g(\beta|a)\, d\beta} = \frac{D^{n-a}}{\Gamma(n-a)} \beta^{a-n-1} e^{-\frac{D}{\beta}}, \qquad \beta > 0, a > 0 \tag{21}$$

2.3 The posterior distribution

The likelihood function can be obtained for the Frechet distribution given in (1) by:

$$L(x_i, \alpha, \beta) = \prod_{i=1}^{n} \alpha \beta^{\alpha} \left(\frac{1}{x_i}\right)^{\alpha+1} e^{-\left(\frac{\beta}{x_i}\right)^{\alpha}} \propto \beta^{n\alpha} e^{-\beta^{\alpha} H} \tag{22}$$

Where

$$H = \sum_{i=1}^{n} \left[\frac{1}{x_i}\right]^{\alpha} \tag{23}$$

The posterior distribution of β can be derived by combining (20) and (22) as follows:

$$h_2(\beta|\underline{x}) = \frac{L(x;\beta) g(\beta|a)}{\int_0^\infty L(x;\beta) g(\beta|a)\, d\beta} = \frac{H^{n+\frac{a}{\alpha}}}{\Gamma(n+\frac{a}{\alpha})} \beta^{n+\frac{a}{\alpha}-1} e^{-\beta H}, \qquad \beta > 0, a > 0 \tag{24}$$

3 The QE-Bayesian Estimation

In this section, we derive the QE-Bayes estimates for scale parameter of the Frechet distribution based on squared error (SELF), weighted balanced (WBLF), entropy (ELF), and minimum expected (MELF) loss functions.

According to Han [21], the hyper parameter a must be chosen to guarantee that $g(\beta|a)$ given in (20) is a decreasing function of β . The derivative of $g(\beta|a)$ with respect to β is

$$\frac{dg(\beta|a)}{d\beta} = (a-1)\beta^{a-2} \tag{25}$$

Note that $a > 0$ and $0 < \beta < 1$ then $a > 1$ result in $\frac{dg(\beta|a)}{d\beta} < 0$, and therefore $g(\beta|a)$ is a decreasing function of β. Consequently, it is more convenient to choose the hyper parameter a under the restriction $1 < a < c$, where c is a given upper bound and a positive constant. Then, we can use the following hyper prior distributions of a introduced by Han [22]:

$$\pi_1(a)=\frac{1}{c-1}, \qquad 1<a<c \tag{26}$$

$$\pi_2(a)=\frac{2a}{c^2-1}, \qquad 1<a<c \tag{27}$$

$$\pi_3(a)=\frac{2(c-a)}{(c-1)^2}, \qquad 1<a<c \tag{28}$$

Reyad et al. [23] defined the QE-Bayes estimatator of β , denoted as $\hat{\beta}_{QEB}$, to be

$$\hat{\beta}_{QEB}=\int_{\Omega}\hat{\beta}_{QB}(a)\,\pi(a)\,da \tag{29}$$

where Ω is domain of a , $\hat{\beta}_{QB}(a)$ is the quasi-Bayes estimation of β with the hyper parameter a and $\pi(a)$ is the hyper prior distribution associated to the hyper parameter a over Ω .

3.1 The QE-Bayesian estimation under squared error loss function (SELF)

Theorem 1. Assuming SELF in (4), the quasi-posterior distribution in (21) and the hyper prior distributions of a in (26), (27) and (28), we have:

(i) The quasi-Bayesian estimate $\hat{\beta}_{QBS}$ of β based on SELF is

$$\hat{\beta}_{QBS}=\frac{D}{n-a-1} \tag{30}$$

(ii) The QE-Bayesian estimates $\hat{\beta}_{QEBS1},\hat{\beta}_{QEBS2}$ and $\hat{\beta}_{QEBS3}$ of β based on $\pi_1(a),\pi_2(a)$ and $\pi_3(a)$ respectively relative to SELF are the following:

$$\hat{\beta}_{QEBS1}=\left(\frac{D}{c-1}\right)\ln\left[1+\frac{c-1}{n-c-1}\right], \tag{31}$$

$$\hat{\beta}_{QEBS2}=\left(\frac{2D}{c+1}\right)\left[\left(\frac{n-1}{c-1}\right)\ln\left(1+\frac{c-1}{n-c-1}\right)-1\right] \tag{32}$$

And

$$\hat{\beta}_{QEBS3}=\left(\frac{2D}{c-1}\right)\left[1-\left(\frac{n-c-1}{c-1}\right)\ln\left(1+\frac{c-1}{n-c-1}\right)\right] \tag{33}$$

Proof. (i) We can obtain the quasi-Bayesian estimate $\hat{\beta}_{QBS}$ by using (5) in (21) as follows:

$$\hat{\beta}_{QBS}=E_{h_1}\left(\beta|X\right)=\int_0^{\infty}\beta\left[\frac{D^{n-a}}{\Gamma(n-a)}\beta^{a-n-1}e^{-\frac{D}{\beta}}\right]d\beta=\frac{D}{n-a-1}$$

where E_{h_1} is the expectation corresponding to quasi-posterior distribution.

(ii) The QE-Bayesian estimate $\hat{\beta}_{QEBS1}$ based on $\pi_1(a)$ can be obtained by using (26) and (30) in (29):

$$\hat{\beta}_{QEBS1} = \int_1^c \left(\frac{D}{n-a-1}\right)\frac{1}{c-1}da = \left(\frac{D}{c-1}\right)\ln\left[1+\frac{c-1}{n-c-1}\right]$$

Similarly, the QE-Bayesian estimates $\hat{\beta}_{QEBS2}, \hat{\beta}_{QEBS3}$ based on $\pi_2(a)$ and $\pi_3(a)$ can be obtained by using (27), (30) in (29) and (28), (30) in (29) respectively:

$$\hat{\beta}_{QEBS2} = \int_1^c \left(\frac{D}{n-a-1}\right)\frac{2a}{c^2-1}da = \left(\frac{2D}{c+1}\right)\left[\left(\frac{n-1}{c-1}\right)\ln\left(1+\frac{c-1}{n-c-1}\right)-1\right]$$

And

$$\hat{\beta}_{QEBS3} = \int_1^c \left(\frac{D}{n-a-1}\right)\frac{2(c-a)}{(c-1)^2}da = \left(\frac{2D}{c-1}\right)\left[1-\left(\frac{n-c-1}{c-1}\right)\ln\left(1+\frac{c-1}{n-c-1}\right)\right]$$

3.2 The QE-Bayesian estimation under weighted balanced loss function (WBLF)

Theorem 2. Assuming WBLF in (6), the quasi-posterior distribution in (21) and the hyper prior distributions of a in (26), (27) and (28), we have:

i) The quasi-Bayesian estimate $\hat{\beta}_{QBW}$ of β based on WBLF is

$$\hat{\beta}_{QBW} = \frac{D}{n-a-2} \tag{34}$$

(ii) The QE-Bayesian estimates $\hat{\beta}_{QEBW1}, \hat{\beta}_{QEBW2}$ and $\hat{\beta}_{QEBW3}$ of β based on $\pi_1(a), \pi_2(a)$ and $\pi_3(a)$ respectively relative to WBLF are the following:

$$\hat{\beta}_{QEBW1} = \left(\frac{D}{c-1}\right)\ln\left[1+\frac{c-1}{n-c-2}\right], \tag{35}$$

$$\hat{\beta}_{QEBW2} = \left(\frac{2D}{c+1}\right)\left[\left(\frac{n-2}{c-1}\right)\ln\left(1+\frac{c-1}{n-c-2}\right)-1\right] \tag{36}$$

And

$$\hat{\beta}_{QEBW3} = \left(\frac{2D}{c-1}\right)\left[1-\left(\frac{n-c-2}{c-1}\right)\ln\left(1+\frac{c-1}{n-c-2}\right)\right] \tag{37}$$

Proof. (i) We can obtain the quasi-Bayesian estimate $\hat{\beta}_{QBW}$ by using (7) in (21) as follows:

$$E_{h_1}\left(\beta^2 \middle| X\right) = \int_0^\infty \beta^2 \left[\frac{D^{n-a}}{\Gamma(n-a)}\beta^{a-n-1} e^{-\frac{D}{\beta}}\right] d\beta = \frac{D^2}{(n-a-1)(n-a-2)}$$

Then, we have

$$\hat{\beta}_{QBW} = \frac{E_{h_1}\left(\beta^2 \middle| X\right)}{E_{h_1}\left(\beta \middle| X\right)} = \frac{D^2/(n-a-1)(n-a-2)}{D/(n-a-1)} = \frac{D}{n-a-2}$$

(ii) The QE-Bayesian estimate $\hat{\beta}_{QEBW\,1}$ based on $\pi_1(a)$ can be obtained by using (26) and (34) in (29):

$$\hat{\beta}_{QEBW\,1} = \int_1^c \left(\frac{D}{n-a-2}\right)\frac{1}{c-1} da = \left(\frac{D}{c-1}\right)\ln\left[1 + \frac{c-1}{n-c-2}\right]$$

Similarly, the QE-Bayesian $\hat{\beta}_{QEBW\,2}, \hat{\beta}_{QEBW\,3}$ based on $\pi_2(a)$ and $\pi_3(a)$ can be obtained by using (27), (34) in (29) and (28), (34) in (29) respectively as follows:

$$\hat{\beta}_{QEBW\,2} = \int_1^c \left(\frac{D}{n-a-2}\right)\frac{2a}{c^2-1} da = \left(\frac{2D}{c+1}\right)\left[\left(\frac{n-2}{c-1}\right)\ln\left(1 + \frac{c-1}{n-c-2}\right) - 1\right]$$

And

$$\hat{\beta}_{QEBW\,3} = \int_1^c \left(\frac{D}{n-a-2}\right)\frac{2(c-a)}{(c-1)^2} da = \left(\frac{2D}{c-1}\right)\left[1 - \left(\frac{n-c-2}{c-1}\right)\ln\left(1 + \frac{c-1}{n-c-2}\right)\right]$$

3.3 The QE-Bayesian estimation under entropy loss function (ELF)

Theorem 3. Assuming ELF in (8), the quasi-posterior distribution in (21) and the hyper prior distributions of a in (26), (27) and (28), we have:

i) The quasi-Bayesian estimate $\hat{\beta}_{QBE}$ of β based on ELF is

$$\hat{\beta}_{QBW} = \frac{D}{n-a} \tag{38}$$

(ii) The QE-Bayesian estimates $\hat{\beta}_{QEBE\,1}, \hat{\beta}_{QEBE\,2}$ and $\hat{\beta}_{QEBE\,3}$ of β based on $\pi_1(a), \pi_2(a)$ and $\pi_3(a)$ respectively relative to ELF are the following:

$$\hat{\beta}_{QEBE\,1} = \left(\frac{D}{c-1}\right)\ln\left[1 + \frac{c-1}{n-c}\right], \tag{39}$$

$$\hat{\beta}_{QEBE\,2} = \left(\frac{2D}{c+1}\right)\left[\left(\frac{n}{c-1}\right)\ln\left(1 + \frac{c-1}{n-c}\right) - 1\right] \tag{40}$$

And

$$\hat{\beta}_{QEB\,3} = \left(\frac{2D}{c-1}\right)\left[1-\left(\frac{n-c}{c-1}\right)\ln\left(1+\frac{c-1}{n-c}\right)\right] \quad (41)$$

Proof. (i) We can obtain the quasi-Bayesian estimate $\hat{\beta}_{QBE}$ by using (9) in (21) as follows:

$$E_{h_1}\left(\beta^{-1}|X\right) = \int_0^\infty \beta^{-1}\left[\frac{D^{n-a}}{\Gamma(n-a)}\beta^{a-n-1}e^{-\frac{D}{\beta}}\right]d\beta = \frac{n-a}{D}$$

Then, we have

$$\hat{\beta}_{QBE} = \left[E_{h_1}\left(\beta^{-1}|X\right)\right]^{-1} = \left[\frac{n-a}{D}\right]^{-1} = \frac{D}{n-a}$$

(ii) The QE-Bayesian estimate $\hat{\beta}_{QEBE\,1}$ based on $\pi_1(a)$ can be obtained by using (26) and (38) in (29):

$$\hat{\beta}_{QEBE\,1} = \int_1^c \left(\frac{D}{n-a}\right)\frac{1}{c-1}da = \left(\frac{D}{c-1}\right)\ln\left[1+\frac{c-1}{n-c}\right]$$

Similarly, the QE-Bayesian $\hat{\beta}_{QEBE\,2}, \hat{\beta}_{QEBE\,3}$ based on $\pi_2(a)$ and $\pi_3(a)$ respectively can be obtained by using (27), (38) in (29) and (28), (38) in (29) respectively as follows:

$$\hat{\beta}_{QEBE\,2} = \int_1^c \left(\frac{D}{n-a}\right)\frac{2a}{c^2-1}da = \left(\frac{2D}{c+1}\right)\left[\left(\frac{n}{c-1}\right)\ln\left(1+\frac{c-1}{n-c}\right)-1\right]$$

And

$$\hat{\beta}_{QEBE\,3} = \int_1^c \left(\frac{D}{n-a}\right)\frac{2(c-a)}{(c-1)^2}da = \left(\frac{2D}{c-1}\right)\left[1-\left(\frac{n-c}{c-1}\right)\ln\left(1+\frac{c-1}{n-c}\right)\right]$$

3.4 The QE-Bayesian estimation under minimum expected loss function (MELF)

Theorem 4. Assuming MELF in (10), the quasi-posterior distribution in (21) and the hyper prior distributions of a in (26), (27) and (28), we have:

i) The quasi-Bayesian estimate $\hat{\beta}_{QBM}$ of β based on MELF is

$$\hat{\beta}_{QBM} = \frac{D}{n-a+1} \quad (42)$$

(ii) The QE-Bayesian estimates $\hat{\beta}_{QEBM\,1}, \hat{\beta}_{QEBM\,2}$ and $\hat{\beta}_{QEBM\,3}$ based on $\pi_1(a), \pi_2(a)$ and $\pi_3(a)$ respectively relative to MELF are the following:

$$\hat{\beta}_{QEBM\,1} = \left(\frac{D}{c-1}\right)\ln\left[1+\frac{c-1}{n-c+1}\right], \quad (43)$$

$$\hat{\beta}_{QEBM\,2}=\left(\frac{2D}{c+1}\right)\left[\left(\frac{n+1}{c-1}\right)\ln\left(1+\frac{c-1}{n-c+1}\right)-1\right] \tag{44}$$

And

$$\hat{\beta}_{QEBM\,3}=\left(\frac{2D}{c-1}\right)\left[1-\left(\frac{n-c+1}{c-1}\right)\ln\left(1+\frac{c-1}{n-c+1}\right)\right] \tag{45}$$

Proof. (i) We can obtain the quasi-Bayesian estimate $\hat{\beta}_{QBM}$ by using (11) in (21) as follows:

$$E_{h_1}\left(\beta^{-2}\left|X\right.\right)=\int_0^{\infty}\beta^{-2}\left[\frac{D^{n-a}}{\Gamma(n-a)}\beta^{a-n-1}e^{-\frac{D}{\beta}}\right]d\beta=\frac{(n-a)(n-a+1)}{D^2}$$

Then, we have

$$\hat{\beta}_{QBM}=\frac{E_{h_1}\left(\beta^{-1}\left|X\right.\right)}{E_{h_1}\left(\beta^{-2}\left|X\right.\right)}=\frac{(n-a)/D}{(n-a)(n-a+1)/D^2}=\frac{D}{n-a+1}$$

(ii) The QE-Bayesian estimate $\hat{\beta}_{QEBM\,1}$ based on $\pi_1(a)$ can be obtained by using (26) and (42) in (29):

$$\hat{\beta}_{QEBM\,1}=\int_1^c\left(\frac{D}{n-a+1}\right)\frac{1}{c-1}da=\left(\frac{D}{c-1}\right)\ln\left[1+\frac{c-1}{n-c+1}\right]$$

Similarly, the QE-Bayesian $\hat{\beta}_{QEBM\,2},\hat{\beta}_{QEBM\,3}$ of β based on $\pi_2(a)$ and $\pi_3(a)$ respectively can be obtained by using (27), (42) in (29) and (28), (42) in (29) respectively as follows:

$$\hat{\beta}_{QEBM\,2}=\int_1^c\left(\frac{D}{n-a+1}\right)\frac{2a}{c^2-1}da=\left(\frac{2D}{c+1}\right)\left[\left(\frac{n+1}{c-1}\right)\ln\left(1+\frac{c-1}{n-c+1}\right)-1\right]$$

And

$$\hat{\beta}_{QEBM\,3}=\int_1^c\left(\frac{D}{n-a+1}\right)\frac{2(c-a)}{(c-1)^2}da=\left(\frac{2D}{c-1}\right)\left[1-\left(\frac{n-c+1}{c-1}\right)\ln\left(1+\frac{c-1}{n-c+1}\right)\right]$$

4 The E-Bayesian Estimation

In this section, we propose the E-Bayes estimates for scale parameter of the Frechet distribution based on squared error (SELF), weighted balanced (WBLF), entropy (ELF), and minimum expected (MELF) loss functions.

4.1 The E-Bayesian estimation under squared error loss function (SELF)

Theorem 5. Assuming SELF in (4), the posterior distribution in (24) and the hyper prior distributions of a in (26), (27) and (28), we have:

(i) The Bayesian estimate $\hat{\beta}_{BS}$ of β based on SELF is

$$\hat{\beta}_{BS} = \frac{n\alpha + a}{\alpha H} \tag{46}$$

(ii) The E-Bayesian estimates $\hat{\beta}_{EBS\,1}, \hat{\beta}_{EBS\,2}$ and $\hat{\beta}_{EBS\,3}$ of β based on $\pi_1(a), \pi_2(a)$ and $\pi_3(a)$ respectively relative to SELF are the following:

$$\hat{\beta}_{EBS\,1} = \frac{2n\alpha + c + 1}{2\alpha H}, \tag{47}$$

$$\hat{\beta}_{EBS\,2} = \frac{3n\alpha(c+1) + 2(c^2 + c + 1)}{3\alpha H\,(c+1)} \tag{48}$$

And

$$\hat{\beta}_{EBS\,3} = \frac{3n\alpha(c-1) + c\,(c+1) - 2}{3\alpha H\,(c-1)} \tag{49}$$

Proof. (i) We can obtain the Bayesian estimate $\hat{\beta}_{BS}$ by using (5) in (24) as follows:

$$\hat{\beta}_{BS} = E_{h_2}\left(\beta \middle| X\right) = \int_0^\infty \beta \left[\frac{H^{n+\frac{a}{\alpha}}}{\Gamma(n + \frac{a}{\alpha})} \beta^{n+\frac{a}{\alpha}-1} e^{-\beta H} \right] d\beta = \frac{n\alpha + a}{\alpha H}$$

where E_{h_2} is the expectation corresponding to posterior distribution.

(ii) The E-Bayesian estimate $\hat{\beta}_{EBS\,1}$ based on $\pi_1(a)$ can be obtained by using (26) and (46):

$$\hat{\beta}_{EBS\,1} = \int_1^c \left(\frac{n\alpha + a}{\alpha H} \right) \frac{1}{c-1} da = \frac{2n\alpha + c + 1}{2\alpha H}$$

Similarly, the E-Bayesian estimates $\hat{\beta}_{EBS\,2}, \hat{\beta}_{EBS\,3}$ of β based on $\pi_2(a)$ and $\pi_3(a)$ can be obtained by using (27), (46) and (28), (46) respectively as follows:

$$\hat{\beta}_{EBS\,2} = \int_1^c \left(\frac{n\alpha + a}{\alpha H} \right) \frac{2a}{c^2 - 1} da = \frac{3n\alpha(c+1) + 2(c^2 + c + 1)}{3\alpha H\,(c+1)}$$

And

$$\hat{\beta}_{EBS\,3} = \int_1^c \left(\frac{n\alpha + a}{\alpha H} \right) \frac{2(c-a)}{(c-1)^2} da = \frac{3n\alpha(c-1) + c\,(c+1) - 2}{3\alpha H\,(c-1)}$$

4.2 The E-Bayesian estimation under weighted balanced loss function (WBLF)

Theorem 6. Assuming WBLF in (6), the posterior distribution in (24) and the hyper prior distributions of a in (26), (27) and (28), we have:

(i) The Bayesian estimate $\hat{\beta}_{BW}$ of β based on WBLF is

$$\hat{\beta}_{BW} = \frac{\alpha(n+1)+a}{\alpha H} \tag{50}$$

(ii) The E-Bayesian estimates $\hat{\beta}_{EBW\,1}, \hat{\beta}_{EBW\,2}$ and $\hat{\beta}_{EBW\,3}$ of β based on $\pi_1(a), \pi_2(a)$ and $\pi_3(a)$ respectively relative to WBLF are the following:

$$\hat{\beta}_{EBW\,1} = \frac{2\alpha(n+1)+c+1}{2\alpha H}, \tag{51}$$

$$\hat{\beta}_{EBW\,2} = \frac{3\alpha(n+1)(c+1)+2(c^2+c+1)}{3\alpha H\,(c+1)} \tag{52}$$

And

$$\hat{\beta}_{EBW\,3} = \frac{3\alpha(n+1)(c-1)+c\,(c+1)-2}{3\alpha H\,(c-1)} \tag{53}$$

Proof. (i) We can obtain the Bayesian estimate $\hat{\beta}_{BW}$ by using (7) in (24) as follows:

$$E_{h_2}\left(\beta^2 \middle| X\right) = \int_0^\infty \beta^2 \left[\frac{H^{n+\frac{a}{\alpha}}}{\Gamma(n+\frac{a}{\alpha})} \beta^{n+\frac{a}{\alpha}-1} e^{-\beta H}\right] d\beta = \frac{(n\alpha+a)(n\alpha+a+\alpha)}{\alpha^2 H^2}$$

Then, we have

$$\hat{\beta}_{BW} = \frac{E_{h_2}\left(\beta^2 \middle| X\right)}{E_{h_2}\left(\beta \middle| X\right)} = \frac{(n\alpha+a)(n\alpha+a+\alpha)/\alpha^2 H^2}{(n\alpha+a)/\alpha H} = \frac{\alpha(n+1)+a}{\alpha H}$$

(ii) The E-Bayesian estimate $\hat{\beta}_{EBW\,1}$ based on $\pi_1(a)$ can be obtained by using (26) and (50) as follows:

$$\hat{\beta}_{EBW\,1} = \int_1^c \left(\frac{\alpha(n+1)+a}{\alpha H}\right)\frac{1}{c-1}da = \frac{2\alpha(n+1)+c+1}{2\alpha H}$$

Similarly, the E-Bayesian $\hat{\beta}_{EBW\,2}, \hat{\beta}_{EBW\,3}$ of β based on $\pi_2(a)$ and $\pi_3(a)$ can be obtained by using (27), (50) and (28), (50) respectively:

$$\hat{\beta}_{EBW\,2} = \int_1^c \left(\frac{\alpha(n+1)+a}{\alpha H}\right)\frac{2a}{c^2-1}da = \frac{3\alpha(n+1)(c+1)+2(c^2+c+1)}{3\alpha H\,(c+1)}$$

And

$$\hat{\beta}_{EBW\,3} = \int_1^c \left(\frac{\alpha(n+1)+a}{\alpha H}\right)\frac{2(c-a)}{(c-1)^2}da = \frac{3\alpha(n+1)(c-1)+c\,(c+1)-2}{3\alpha H\,(c-1)}$$

4.3 The E-Bayesian estimation under entropy loss function (ELF)

Theorem 7. Assuming ELF in (8), the posterior distribution in (24) and the hyper prior distributions of a in (26), (27) and (28), we have:

i) The Bayesian estimate $\hat{\beta}_{BE}$ of β based on ELF is

$$\hat{\beta}_{BE} = \frac{\alpha(n-1)+a}{\alpha H} \tag{54}$$

(ii) The E-Bayesian estimates $\hat{\beta}_{EBE\,1}, \hat{\beta}_{EBE\,2}$ and $\hat{\beta}_{EBE\,3}$ of β based on $\pi_1(a), \pi_2(a)$ and $\pi_3(a)$ respectively relative to ELF are the following:

$$\hat{\beta}_{EBE\,1} = \frac{2\alpha(n-1)+c+1}{2\alpha H}, \tag{55}$$

$$\hat{\beta}_{EBE\,2} = \frac{3\alpha(n-1)(c+1)+2(c^2+c+1)}{3\alpha H\,(c+1)} \tag{56}$$

And

$$\hat{\beta}_{EBE\,3} = \frac{3\alpha(n-1)(c-1)+c\,(c+1)-2}{3\alpha H\,(c-1)} \tag{57}$$

Proof. (i) We can obtain the Bayesian estimate $\hat{\beta}_{BE}$ by using (9) in (24) as follows:

$$E_{h_2}\left(\beta^{-1}\middle|X\right) = \int_0^\infty \beta^{-1}\left[\frac{H^{n+\frac{a}{\alpha}}}{\Gamma(n+\frac{a}{\alpha})}\beta^{n+\frac{a}{\alpha}-1}e^{-\beta H}\right]d\beta = \frac{\alpha H}{\alpha(n-1)+a}$$

Then, we have

$$\hat{\beta}_{BE} = \left[E_{h_2}\left(\beta^{-1}\middle|X\right)\right]^{-1} = \left[\frac{\alpha H}{\alpha(n-1)+a}\right]^{-1} = \frac{\alpha(n-1)+a}{\alpha H}$$

(ii) The E-Bayesian estimate $\hat{\beta}_{EBE\,1}$ based on $\pi_1(a)$ can be obtained by using (26) and (54):

$$\hat{\beta}_{EBE\,1} = \int_1^c \left(\frac{\alpha(n-1)+a}{\alpha H}\right)\frac{1}{c-1}da = \frac{2\alpha(n-1)+c+1}{2\alpha H}$$

Similarly, the E-Bayesian estimates $\hat{\beta}_{EBE\,2}, \hat{\beta}_{EBE\,3}$ of β based on $\pi_2(a)$ and $\pi_3(a)$ can be obtained by using (27), (54) and (28), (54) respectively as follows:

$$\hat{\beta}_{EBE\,2} = \int_1^c \left(\frac{\alpha(n-1)+a}{\alpha H}\right)\frac{2a}{c^2-1}da = \frac{3\alpha(n-1)(c+1)+2(c^2+c+1)}{3\alpha H\,(c+1)}$$

And

$$\hat{\beta}_{EBE\,3}=\int_1^c\left(\frac{\alpha(n-1)+a}{\alpha H}\right)\frac{2(c-a)}{(c-1)^2}da=\frac{3\alpha(n-1)(c-1)+c(c+1)-2}{3\alpha H\,(c-1)}$$

4.4 The E-Bayesian estimation under minimum expected loss function (MELF)

Theorem 8. Assuming MELF in (10), the posterior distribution in (24) and the hyper prior distributions of a in (26), (27) and (28), we have:

i) The Bayesian estimate $\hat{\beta}_{BM}$ of β based on MELF is

$$\hat{\beta}_{BM}=\frac{\alpha(n-2)+a}{\alpha H} \tag{58}$$

(ii) The E-Bayesian estimates $\hat{\beta}_{EBM\,1},\hat{\beta}_{EBM\,2}$ and $\hat{\beta}_{EBM\,3}$ of β based on $\pi_1(a),\pi_2(a)$ and $\pi_3(a)$ respectively relative to MELF are the following:

$$\hat{\beta}_{EBM\,1}=\frac{2\alpha(n-2)+c+1}{2\alpha H}, \tag{59}$$

$$\hat{\beta}_{EBM\,2}=\frac{3\alpha(n-2)(c+1)+2(c^2+c+1)}{3\alpha H\,(c+1)} \tag{60}$$

And

$$\hat{\beta}_{EBM\,3}=\frac{3\alpha(n-2)(c-1)+c(c+1)-2}{3\alpha H\,(c-1)} \tag{61}$$

Proof. (i) We can obtain the Bayesian estimate $\hat{\beta}_{BM}$ by using (11) in (24) as follows:

$$E_{h_2}\left(\beta^{-2}\left|X\right.\right)=\int_0^\infty\beta^{-2}\left[\frac{H^{n+\frac{a}{\alpha}}}{\Gamma(n+\dfrac{a}{\alpha})}\beta^{n+\frac{a}{\alpha}-1}e^{-\beta H}\right]d\beta=\frac{\alpha^2H^2}{(n\alpha+a-\alpha)(n\alpha+a-2\alpha)}$$

Then, we have

$$\hat{\beta}_{BM}=\frac{E_{h_2}\left(\beta^{-1}\left|X\right.\right)}{E_{h_2}\left(\beta^{-2}\left|X\right.\right)}=\frac{\alpha H/(n\alpha+a-\alpha)}{\alpha^2H^2/(n\alpha+a-\alpha)(n\alpha+a-2\alpha)}=\frac{\alpha(n-2)+a}{\alpha H}$$

(ii) The E-Bayesian estimate $\hat{\beta}_{EBM\,1}$ based on $\pi_1(a)$ can be obtained by using (26) and (58) as follows:

$$\hat{\beta}_{EBM\,1}=\int_1^c\left(\frac{\alpha(n-2)+a}{\alpha H}\right)\frac{1}{c-1}da=\frac{2\alpha(n-2)+c+1}{2\alpha H}$$

Similarly, the E-Bayesian $\hat{\beta}_{EBM\,2},\hat{\beta}_{EBM\,3}$ of β based on $\pi_2(a)$ and $\pi_3(a)$ can be obtained by using (27), (58) and (28), (58) respectively as follows:

$$\hat{\beta}_{EBM\,2} = \int_1^c \left(\frac{\alpha(n-2)+a}{\alpha H}\right)\frac{2a}{c^2-1}da = \frac{3\alpha(n-2)(c+1)+2(c^2+c+1)}{3\alpha H\,(c+1)}$$

And

$$\hat{\beta}_{EBM\,3} = \int_1^c \left(\frac{\alpha(n-2)+a}{\alpha H}\right)\frac{2(c-a)}{(c-1)^2}da = \frac{3\alpha(n-2)(c-1)+c\,(c+1)-2}{3\alpha H\,(c-1)}$$

5 Properties of the QE-Bayesian and E- Bayesian Estimates

In this section, we investigate the properties of the QE-Bayesian and the E-Bayesian estimates.

5.1 The relations between the QE-Bayesian estimates

In this subsection, we derive the relations among the QE-Bayesian estimates.

5.1.1 Relations among $\hat{\beta}_{QEBSi}\ (i=1,2,3)$

Theorem 9. According to (31), (32) and (33) we have:

(i) $\hat{\beta}_{QEBS\,3} < \hat{\beta}_{QEBS\,1} < \hat{\beta}_{QEBS\,2}$ (ii) $\lim_{n\to\infty}\hat{\beta}_{QEBS\,1} = \lim_{n\to\infty}\hat{\beta}_{QEBS\,2} = \lim_{n\to\infty}\hat{\beta}_{QEBS\,3}$

Proof. See Appendix (1).

5.1.2 Relations among $\hat{\beta}_{QEBWi}\ (i=1,2,3)$

Theorem 10. From (35), (36) and (37) we obtain:

(i) $\hat{\beta}_{QEBW\,3} < \hat{\beta}_{QEBW\,1} < \hat{\beta}_{QEBW\,2}$ (ii) $\lim_{n\to\infty}\hat{\beta}_{QEBW\,1} = \lim_{n\to\infty}\hat{\beta}_{QEBW\,2} = \lim_{n\to\infty}\hat{\beta}_{QEBW\,3}$

Proof. See Appendix (1).

5.1.3 Relations among $\hat{\beta}_{QEBEi}\ (i=1,2,3)$

Theorem 11. Using (39), (40) and (41) we have:

(i) $\hat{\beta}_{QEBE\,3} < \hat{\beta}_{QEBE\,1} < \hat{\beta}_{QEBE\,2}$ (ii) $\lim_{n\to\infty}\hat{\beta}_{QEBE\,1} = \lim_{n\to\infty}\hat{\beta}_{QEBE\,2} = \lim_{n\to\infty}\hat{\beta}_{QEBE\,3}$

Proof. See Appendix (1).

5.1.4 Relations among $\hat{\beta}_{QEBMi}\ (i=1,2,3)$

Theorem 12. From (43), (44) and (45) we have:

(i) $\hat{\beta}_{QEBM\,3} < \hat{\beta}_{QEBM\,1} < \hat{\beta}_{QEBM\,2}$ (ii) $\lim_{n\to\infty}\hat{\beta}_{QEBM\,1} = \lim_{n\to\infty}\hat{\beta}_{QEBM\,2} = \lim_{n\to\infty}\hat{\beta}_{QEBM\,3}$

Proof. See Appendix (1).

5.2 The relations between the E-Bayesian estimates

In this subsection, we show the properties of the E-Bayesian estimates.

5.2.1 Relations among $\hat{\beta}_{EBSi}\ (i = 1,2,3)$

Theorem 13. Using (47), (48) and (49) we have:

(i) $\hat{\beta}_{EBS\,3} < \hat{\beta}_{EBS\,1} < \hat{\beta}_{EBS\,2}$ (ii) $\lim_{H\to\infty}\hat{\beta}_{EBS\,1} = \lim_{H\to\infty}\hat{\beta}_{EBS\,2} = \lim_{H\to\infty}\hat{\beta}_{EBS\,3}$

Proof. See Appendix (2).

5.2.2 Relations among $\hat{\beta}_{EBWi}\ (i = 1,2,3)$

Theorem 14. From (51), (52) and (53) we obtain:

(i) $\hat{\beta}_{EBW\,3} < \hat{\beta}_{EBW\,1} < \hat{\beta}_{EBW\,2}$ (ii) $\lim_{H\to\infty}\hat{\beta}_{EBW\,1} = \lim_{H\to\infty}\hat{\beta}_{EBW\,2} = \lim_{H\to\infty}\hat{\beta}_{EBW\,3}$

Proof. See Appendix (2).

5.2.3 Relations among $\hat{\beta}_{EBEi}\ (i = 1,2,3)$

Theorem 15 According to (55), (56) and (57) we have:

(i) $\hat{\beta}_{EBE\,3} < \hat{\beta}_{EBE\,1} < \hat{\beta}_{EBE\,2}$ (ii) $\lim_{H\to\infty}\hat{\beta}_{EBE\,1} = \lim_{H\to\infty}\hat{\beta}_{EBE\,2} = \lim_{H\to\infty}\hat{\beta}_{EBE\,3}$

Proof. See Appendix (2).

5.2.4 Relations among $\hat{\beta}_{EBMi}\ (i = 1,2,3)$

Theorem 16. From (59), (60) and (61) we have :

(i) $\hat{\beta}_{EBM\,3} < \hat{\beta}_{EBM\,1} < \hat{\beta}_{EBM\,2}$ (ii) $\lim_{H\to\infty}\hat{\beta}_{EBM\,1} = \lim_{H\to\infty}\hat{\beta}_{EBM\,2} = \lim_{H\to\infty}\hat{\beta}_{EBM\,3}$

Proof. See Appendix (2).

6 Monte Carlo Simulation

In this section a Monte Carlo simulation is performed to assess the performance of the QE-Bayes and E-Bayes estimates associated to scale parameter of the Ftechet distribution discussed in the previous sections. The simulation structure can be described in the following steps:

Step (1): Set the default combinations (true values) of α and c which are $(\alpha = 3, c = 3), (\alpha = 4, c = 3)$ and $(\alpha = 3, c = 4)$ respectively. We considered different sample sizes to study their effect on the resulting estimates.

Step (2): For these cases, we generate a from the uniform prior distributions $(1, c)$ given in (26), (27) and (28). For given values of a , we generate β from the power density given in (20).

Step (3): For known values of α , samples are generated from the Frechet distribution given in (1).

Step (4): Calculate the QE-Bayes and E-Bayes estimates of scale parameter associated to the Frechet distribution according to formulas that have been derived.

Step (5): We repeated this process 10000 times and compute the absolute bias (ABias) and mean square error (MSE) for the estimates for different sample sizes and given values of α, c where

$ABias = \left|\hat{\beta} - \beta\right|$ $\quad MSE(\hat{\theta}) = \frac{1}{10000}\sum(\hat{\beta} - \beta)^2$ and

$\hat{\beta}$ stands for an estimator of β . The simulation results are shown in Tables 1-4.

Table 1. Average values of ABias and MSEs (within parenthesis) for estimates of the parameter β based on SELF

Sample	QE-Bayes			E-Bayes		
n	$c=3,\ \alpha=3$	$c=4,\ \alpha=3$	$c=3,\ \alpha=4$	$c=3,\ \alpha=3$	$c=4,\ \alpha=3$	$c=3,\ \alpha=4$
15	0.1684913	0.2234576	0.1691563	0.3341981	0.3014775	0.447804
	(0.0477689)	(0.0774759)	(0.0371641)	(0.1201167)	(0.1041075)	(0.2042198)
	0.1800712	0.2480614	0.180813	0.3330296	0.2988633	0.4472106
	(0.0523483)	(0.0905395)	(0.0414832)	(0.1193969)	(0.1027056)	(0.2037076)
	0.1453315	0.1824512	0.1458431	0.3365351	0.3058345	0.4489909
	(0.0394372)	(0.0584776)	(0.0293512)	(0.1215665)	(0.1064767)	(0.2052437)
25	0.0910873	0.1189299	0.0918647	0.3503312	0.3215957	0.455914
	(0.0182147)	(0.026802)	(0.012699)	(0.1269499)	(0.1101182)	(0.2098973)
	0.0968086	0.1305868	0.0976287	0.3496529	0.3200714	0.4555665
	(0.0194406)	(0.0300668)	(0.013856)	(0.1264933)	(0.109192)	(0.2095873)
	0.0796446	0.0995019	0.0803367	0.3516879	0.3241364	0.4566089
	(0.0159626)	(0.0219757)	(0.0105856)	(0.1278658)	(0.1116726)	(0.2105181)
35	0.0617829	0.0817595	0.0628113	0.3576093	0.3283129	0.4597598
	(0.0105762)	(0.0151023)	(0.0066201)	(0.1306952)	(0.1123406)	(0.2126437)
	0.0655625	0.089356	0.0662152	0.3571325	0.3272327	0.4595147
	(0.0111282)	(0.0165633)	(0.00714122)	(0.1303631)	(0.1116579)	(0.2124214)
	0.0542237	0.0690987	0.0551908	0.3585629	0.3301133	0.4602499
	(0.00955921)	(0.0129273)	(0.0056654)	(0.1313606)	(0.1134837)	(0.2130887)
50	0.0435082	0.0539605	0.0423655	0.3610573	0.3351018	0.4628626
	(0.0063472)	(0.0081410)	(0.0036152)	(0.1322711)	(0.1153076)	(0.2151287)
	0.0460164	0.0589255	0.0448868	0.3607254	0.3343537	0.4626929
	(0.0066034)	(0.0087691)	(0.0038481)	(0.1320357)	(0.1148186)	(0.2149731)
	0.0384919	0.0456856	0.0373228	0.361721	0.3363487	0.4632021
	(0.0058729)	(0.0072045)	(0.0031876)	(0.1327424)	(0.1161252)	(0.2154401)
70	0.0305688	0.0378697	0.0297147	0.3644888	0.3399208	0.4647774
	(0.0040892)	(0.005006)	(0.0021313)	(0.1341684)	(0.1176508)	(0.2166191)
	0.032296	0.0412683	0.0314518	0.3642536	0.3393908	0.4646569
	(0.0042135)	(0.0053073)	(0.0022438)	(0.1339992)	(0.1172967)	(0.2165079)
	0.0271145	0.0322053	0.0262405	0.3649593	0.3408043	0.4650182
	(0.0038586)	(0.0045555)	(0.0019245)	(0.1345074)	(0.1182423)	(0.2168417)
100	0.0204802	0.0258168	0.0208214	0.3666277	0.3427861	0.4659545
	(0.0025558)	(0.0034437)	(0.0012928)	(0.1353307)	(0.1189337)	(0.2175354)
	0.0216559	0.0281212	0.0220056	0.3664637	0.3424165	0.4658705
	(0.0026127)	(0.0035852)	(0.0013465)	(0.1352114)	(0.1186833)	(0.2174575)
	0.0181294	0.0219761	0.0184519	0.3669556	0.3434019	0.4661225
	(0.0024503)	(0.0032314)	(0.0011939)	(0.1355692)	(0.1193516)	(0.2176913)

Table 2. Average values of ABias and MSEs (within parenthesis) for estimates of the parameter β based on WBLF

Sample	QE-Bayes			E-Bayes		
n	$c=3,\ \alpha=3$	$c=4,\ \alpha=3$	$c=3,\ \alpha=4$	$c=3,\ \alpha=3$	$c=4,\ \alpha=3$	$c=3,\ \alpha=4$
15	0.2445485	0.2118284	0.2457175	0.3131652	0.2944054	0.4435615
	(0.0828877)	(0.0644792)	(0.0705616)	(0.1076112)	(0.0965068)	(0.1921575)
	0.2583405	0.2298192	0.2596009	0.3119967	0.2924772	0.4329681
	(0.0905312)	(0.0731933)	(0.0778888)	(0.1069442)	(0.0954655)	(0.1916645)
	0.2169645	0.1818427	0.2179507	0.3155022	0.2976193	0.4347484
	(0.0687765)	(0.0514272)	(0.0570778)	(0.1089535)	(0.0982598)	(0.1931462)
25	0.1270834	0.1597713	0.1281292	0.3381215	0.3063521	0.4475748
	(0.0270365)	(0.0394544)	(0.0210929)	(0.1188791)	(0.1010742)	(0.20252627)
	0.1333633	0.1725966	0.1344559	0.3374432	(0.3048277	0.4472273
	(0.0288457)	(0.0441281)	(0.0228288)	(0.1184399)	(0.1001965)	(0.2022221)
	0.1145234	0.1383951	0.1154756	0.3394782	0.3088927	0.4482697
	(0.0236589)	(0.0324084)	(0.0178631)	(0.1197605)	(0.1025479)	(0.2031348)
35	0.085202	0.1079372	0.0864201	0.3490269	0.3175108	0.4538784
	(0.0144619)	(0.020634)	(0.0103188)	(0.1247904)	(0.1056218)	(0.2073421)
	0.0892295	0.1160407	0.0904803	0.3485501	0.3164305	0.4536329
	(0.0152421)	(0.0226226)	(0.0110678)	(0.1244668)	(0.1049632)	(0.2071228)
	0.0771469	0.0944313	0.0782997	0.3499805	0.3193111	0.4543682
	(0.0130001)	(0.0176046)	(0.0089202)	(0.1254391)	(0.1067249)	(0.2077812)
50	0.0588874	0.0708815	0.0578253	0.3550838	0.3276207	0.4587893
	(0.0081178)	(0.0104859)	(0.0052442)	(0.1280693)	(0.1104694)	(0.2114122)
	0.0615059	0.0760672	0.0604574	0.3547519	0.3268726	0.4586196
	(0.0084668)	(0.0113203)	(0.0055693)	(0.1278384)	(0.1099919)	(0.2112558)
	0.0536505	0.0622386	0.0525609	0.3557475	0.3288676	0.4591288
	(0.0074613)	(0.0092160)	(0.0046356)	(0.1285326)	(0.1112679)	(0.2117186)
70	0.0410899	0.0493738	0.0402963	0.3602545	0.3346201	0.4618868
	(0.0049395)	(0.0061196)	(0.0029102)	(0.1311371)	(0.1141351)	(0.2139577)
	0.0428698	0.0528771	0.0420864	0.3600192	0.3340924	0.4617663
	(0.0051054)	(0.0065116)	(0.0030642)	(0.1309698)	(0.1137866)	(0.2138472)
	0.0375314	0.0435351	0.0367163	0.3607254	0.3355035	0.4621277
	(0.0046268)	(0.0055211)	(0.0026215)	(0.1314721)	(0.1147171)	(0.2141788)
100	0.0276078	0.0335794	0.0280026	0.3636765	0.3390907	0.4639385
	(0.0029432)	(0.0039632)	(0.0016614)	(0.1331936)	(0.1164425)	(0.2156692)
	0.028808	0.0539323	0.0292124	0.3635125	0.3387212	0.4638545
	(0.0030185)	(0.0041446)	(0.0017337)	(0.1330754)	(0.1161949)	(0.2155916)
	0.0252073	0.0296578	0.0255839	0.3640044	0.3397066	0.4641065
	(0.0028014)	(0.0036856)	(0.0015257)	(0.1334302)	(0.1168558)	(0.2158244)

Table 3. Average values of ABias and MSEs (within parenthesis) for estimates of the parameter β based on ELF

Sample	QE-Bayes			E-Bayes		
n	$c=3,\ \alpha=3$	$c=4,\ \alpha=3$	$c=3,\ \alpha=4$	$c=3,\ \alpha=3$	$c=4,\ \alpha=3$	$c=3,\ \alpha=4$
15	0.1042111	0.1053038	0.1044502	0.3552311	0.3329689	0.4620465
	(0.0273614)	(0.0264504)	(0.0181903)	(0.1335757)	(0.1189818)	(0.2167166)
	0.1140718	0.1194005	0.1143763	0.3540625	0.3310408	0.4614531
	(0.0299403)	(0.0301503)	(0.0205505)	(0.1328034)	(0.1177835)	(0.2161871)
	0.0844895	0.0818092	0.0845979	0.3575681	0.3361826	0.4632334
	(0.0228032)	(0.0211873)	(0.0140681)	(0.1351301)	(0.1209964)	(0.2177778)

Sample	**QE-Bayes**			**E-Bayes**		
n	$c=3,\ \alpha=3$	$c=4,\ \alpha=3$	$c=3,\ \alpha=4$	$c=3,\ \alpha=3$	$c=4,\ \alpha=3$	$c=3,\ \alpha=4$
25	0.0582275	0.0817361	0.0587642	0.3625409	0.3368394	0.4642531
	(0.0124635)	(0.0182351)	(0.0073497)	(0.1353316)	(0.1196469)	(0.2174139)
	0.0634616	0.0923771	0.0640331	0.3618626	0.3353151	0.4639057
	(0.0132325)	(0.0203984)	(0.0080540)	(0.1348577)	0.1186722)	(0.2170978)
	0.0477593	0.0640011	0.0482137	0.3638976	0.3393804	0.4649481
	(0.0110928)	(0.0151424)	(0.0061094)	(0.1362821)	(0.1212822)	(0.2180468)
35	0.0397845	0.0571964	0.0406346	0.3661916	0.3391151	0.4656416
	(0.0079382)	(0.0111783)	(0.00416628)	(0.1367517)	(0.1192998)	(0.2180165)
	0.0433383	0.0643321	0.0442173	0.3657148	0.3380349	0.4653965
	(0.0082980)	(0.0121922)	(0.0044958)	(0.1364112)	(0.1185932)	(0.2177912)
	0.0326767	0.0453037	0.0334692	0.3671452	0.34091554	0.4661317
	(0.0072954)	(0.0097177)	(0.0035847)	(0.1374341)	(0.1204834)	(0.2184674)
50	0.0287701	0.0377528	0.0275501	0.3670308	0.3425829	0.4669359
	(0.0050982)	(0.0064365)	(0.0025043)	(0.1365457)	(0.1202614)	(0.2188812)
	0.0311749	0.0425107	0.0299675	0.3666989	0.3418348	0.4667662
	(0.00527210)	(0.0068819)	(0.0026554)	(0.1363063)	(0.1197596)	(0.2187242)
	0.0239607	0.0298228	0.0227154	0.3676945	0.3438298	0.4727534
	(0.0047856)	(0.0057956)	(0.0022371)	(0.1370251)	(0.1210965)	(0.2191955)
70	0.0203573	0.0267065	0.0194444	0.3687231	0.3452216	0.4676679
	(0.0034771)	(0.0041801)	(0.0015899)	(0.137236)	(0.1212236)	(0.2192975)
	0.0220341	0.0300052	0.0211308	0.3684879	0.3446915	0.4675475
	(0.0035632)	(0.0043980)	(0.0016643)	(0.1370646)	(0.1208637)	(0.2191855)
	0.0170038	0.0212088	0.0160717	0.3691936	0.3461051	0.4679088
	(0.0033218)	(0.0038654)	(0.00145831)	(0.1375791)	(0.1218246)	(0.2195214)
100	0.0134982	0.0182135	0.0137862	0.3695788	0.3464812	0.4679705
	(0.00227514)	(0.0030522)	(0.0010319)	(0.1374852)	(0.1214525)	(0.2194099)
	0.0146499	0.0204708	0.0149465	0.3694149	0.3461117	0.4678865
	(0.0023147)	(0.0031563)	(0.0010681)	(0.1373651)	(0.1211993)	(0.2193316)
	0.0111947	0.0144511	0.0114652	0.3699067	0.3470971	0.4681385
	(0.0022040)	(0.0029015)	(0.0096755)	(0.1377257)	(0.1218749)	(0.2195664)

Table 4. Average values of ABias and MSEs (within parenthesis) for estimates of the parameter β based on MELF

Sample	**QE-Bayes**			**E-Bayes**		
n	$c=3,\ \alpha=3$	$c=4,\ \alpha=3$	$c=3,\ \alpha=4$	$c=3,\ \alpha=3$	$c=4,\ \alpha=3$	$c=3,\ \alpha=4$
15	0.0491647	0.0607407	0.0490391	0.3762639	0.3522507	0.4762891
	(0.0166377)	(0.0174281)	(0.0086789)	(0.1479883)	(0.1313965)	(0.2296507)
	0.0576629	0.0733488	0.0575935	0.3750954	0.3503225	0.4756956
	(0.0178866)	(0.0195686)	(0.0097416)	(0.1471626)	(0.1301197)	(0.2291031)
	0.0321682	0.0397273	0.0319299	0.3786009	0.3554644	0.4774759
	(0.0145853)	(0.0145834)	(0.0069980)	(0.1496485)	(0.1335419)	(0.2307483)
25	0.0281114	0.0477213	0.0284189	0.3747506	0.3520831	0.4725923
	(0.0091221)	(0.0128666)	(0.0043863)	(0.1440242)	(0.1296605)	(0.2250759)
	0.0329175	0.0574738	0.0332613	0.3740723	0.3505588	0.4722448
	(0.0095316)	(0.0141649)	(0.00473487)	(0.1435331)	(0.1286373)	(0.2247537)
	0.0184978	0.0314672	0.0187342	0.3761073	0.3546237	0.4732872
	(0.0084441)	(0.0111331)	(0.0038309)	(0.1450093)	(0.1313765)	(0.2257209)
35	0.0190812	0.0341027	0.0197636	0.374774	0.3499176	0.4715234
	(0.0063509)	(0.0086042)	(0.0027598)	(0.1429599)	(0.1264994)	(0.2234604)
	0.0224294	0.0408182	0.0231386	0.3742972	0.3488371	0.4712783
	(0.0065488)	(0.0092413)	(0.00292793)	(0.1426111)	(0.1257687)	(0.2232322)

Sample	QE-Bayes			E-Bayes		
n	$c=3,\ \alpha=3$	$c=4,\ \alpha=3$	$c=3,\ \alpha=4$	$c=3,\ \alpha=3$	$c=4,\ \alpha=3$	$c=3,\ \alpha=4$
	0.0123856	0.0229103	0.0130138	0.3757276	0.3517177	0.4720135
	(0.0060234)	(0.0077455)	(0.0024924)	(0.01436591)	(0.1277228)	(0.2239173)
50	0.0146339	0.0222144	0.0133398	0.3730042	0.3500644	0.4710092
	(0.004312090)	(0.0053000)	(0.0018527)	(0.1408942)	(0.1253266)	(0.2226677)
	0.0169414	0.0267777	0.0156594	0.3726724	0.3493159	0.4708394
	(0.00441287)	(0.0055833)	(0.0019314)	(0.1406497)	(0.1248148)	(0.2225092)
	0.0100187	0.0146079	0.0087004	0.3736681	0.3513109	0.4713486
	(0.004142744)	(0.0049213)	(0.0017277)	(0.1413807)	(0.1261822)	(0.2229847)
70	0.0093626	0.0158695	0.0094719	0.3727518	0.3505223	0.4705585
	(0.0027891)	(0.0036181)	(0.0012670)	(0.1401996)	(0.1248534)	(0.2219928)
	0.0109885	0.0190725	0.0111098	0.3725164	0.3499923	0.4704381
	(0.00283526)	(0.0037596)	(0.0013061)	(0.1400262)	(0.1244878)	(0.2218802)
	0.0061109	0.0105312	0.0061962	0.3732226	0.3514058	0.4707994
	(0.0027128)	(0.0034282)	(0.0012040)	(0.1405468)	(0.1254639)	(0.2222182)
100	0.00665719	0.0107645	0.0068931	0.3725314	0.3501765	0.4699865
	(0.0020952)	(0.0027814)	(0.0008724)	(0.1396574)	(0.1239988)	(0.2212925)
	0.0077858	0.0129763	0.0080303	0.3723661	0.3498071	0.4699025
	(0.0021184)	(0.0028501)	(0.0008922)	(0.1395363)	(0.1237429)	(0.2212139)
	0.0043999	0.0070782	0.0046188	0.3728579	0.3507924	0.4701545
	(0.0025644)	(0.0026886)	(0.0008407)	(0.1398999)	(0.1244259)	(0.2214498)

7 Conclusion

1- From simulation study, it is concluded that QE-Bayes estimates performs better than E-Bayes estimates. That means, the QE-Bayes estimates have minimum ABias and MSE as compared with the E-Bayes estimates based on different loss functions and by considering various cominations of n,α and c .

2- By comparing the QE-Bayes estimates based on the different loss functions, we can conclude that the QE-Bayes estimates based on MELF are the most efficient whereas the QE-Bayes estimates based on WBLF are the least efficient in all cases.

3- Furthermore, by observing the QE-Bayes estimates based on different combinations of α,c , we can deducte that the best results are obtained when $\alpha > c$ whereas the worst results are obtained when $\alpha < c$ for all loss functions.

From the previous discussion we conclude that the suggested criteria yield more efficient estimators as compared with the original E-Bayes method and is easy to perform.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Frechet M. Sur la loi de probabilite de lecart maximum. Ann. Soc. Polon. Math. 1927;6:93.

[2] Mubarak M. Parameter estimation based on the Frechet progressive type II censored data with binomial removals. International Journal of Quality, Statistics and Reliability; 2012. Article ID 245910.

[3] Abbas K, Yincai T. Comparison of estimation methods for Frechet distribution with known shape. Caspian Journal of Applied Sciences Research. 2012;1(10):58-64.

[4] Nasir W, Aslam M. Bayes approach to study shape parameter of Frechet distribution. International Journal of Basic Applied Sciences. 2015;4(3):246-254.

[5] Wedderbuem RWM. Quasi-likelihood functions, generalized models and the Gauss-Newton method. Biometrika. 1974;61(3):439-443.

[6] Han M. Expected Bayesian method for forecast of security investment. Journal of Operations Research and Management Science. 2005;14(5):89-102.

[7] Han M. E-Bayesian method to estimate failure rate. The Sixth International Symposium on Operations Research and Its Applications (ISOR06) Xinjiang. 2006;299-311.

[8] Yin Q, Liu H. Bayesian estimation of geometric distribution parameter under scaled squared error loss function. Conference on Environmental Science and Information Application Technology. 2010;650-653.

[9] Wei J, Song B, Yan W, Mao Z. Reliability Estimations of Burr-XII distribution under entropy loss function. IEEE. 2011;244-247.

[10] Jaheen ZF, Okasha HM. E-Bayesian estimation for the Burr type XII model based on type-2 censoring. Applied Mathematical Modelling. 2011;35:4730-4737.

[11] Cai G, Xu W, Zhang W, Wang P. Application of E-Bayes method in stock forecast. Fourth International Conference on Information and Computing. 2011;504-506.

[12] Okasha HM. E-Bayesian estimation of system reliability with Weibull distribution of components based on type-2 censoring. Journal of Advanced Research in Scientific Computing. 2012,4(4):34-45.

[13] Azimi R, Yaghamei F, Fasihi B. E-Bayesian estimation based on generalized half Logistic progressive type-II censored data. International Journal of Advanced Mathematical Science. 2013;1(2):56-63.

[14] Javadkani N, Azhdari P, Azimi R. On Bayesian estimation from two parameter Bathtub-shaped lifetime distribution based on progressive first-failure-censored sampling. International Journal of Scientific World. 2014;2(1):31-41.

[15] Liu H, Yin T, Wang C. E-Bayes estimation of Rayleigh distribution parameter. Journal of Applied Mechanics and Materials. 2014;596:305-308.

[16] Reyad HM, Ahmed SO. E-Bayesian analysis of the Gumbel type-ii distribution under type-ii censored scheme. International Journal of Advanced Mathematical Sciences. 2015;3(2):108-120.

[17] Reyad HM, Ahmed SO. Bayesian and E-Bayesian estimation for the Kumaraswamy distribution based on type-ii censoring. International Journal of Advanced Mathematical Sciences. 2016;4(1): 10-17.

[18] Reyad HM, Younis AM, Alkhedir AA. Comparison of estimates using censored samples from Gompertz model: Bayesian, E-Bayesian, Hierarchical Bayesian and Empirical Bayesian Schemes. International Journal of Advanced Statistics and Probability. 2016;4(1):47-61.

[19] Dey DK, Gosh M, Srinivasan C. Simultaneous estimation of parameter under entropy loss. Journal of Statistical Planning and Inference.1987;15:347-363.

[20] Tummala VM, Sathe PT. Minimum exprcted loss estimators of reliability and parameters of certain lifetime distributions. Reliability, IEEE Transactions. 1978;27(4):283-285.

[21] Han M. The structure of hierarchical prior distribution and its applications. Chinese Operations Research and Management Science. 1997;6(3):31-40.

[22] Han M. E-Bayesian estimation of the reliability derived from Binomial distribution. Applied Mathematical Modelling. 2011;35:2419-2424.

[23] Reyad HM, Younis AM, Alkhedir AA. Quasi-E-Bayesian criteria versus quasi-Bayesian, quasi-hierarchical Bayesian and quasi-empirical Bayesian methods for estimating the scale parameter of the Erlangen distribution. International Journal of Advanced Statistics and Probability. 2016;4(1):62-74.

APPENDIX - 1

Proof of Theorem 9.

(i) From (31), (32) and (33), we get

$$\hat{\beta}_{QEBS\,1}-\hat{\beta}_{QEBS\,3}=\left(\frac{D}{c-1}\right)\left\{\left[1+\frac{2(n-c-1)}{c-1}\right]\ln\left[1+\frac{c-1}{n-c-1}\right]-2\right\} \tag{A.1}$$

$$\hat{\beta}_{QEBS\,2}-\hat{\beta}_{QEBS\,1}=\left(\frac{D}{c+1}\right)\left\{\left[1+\frac{2(n-c-1)}{c-1}\right]\ln\left[1+\frac{c-1}{n-c-1}\right]-2\right\} \tag{A.2}$$

For $-1<x<1,$ we have: $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+...=\sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k}.$

Assuming $x=\frac{c-1}{n-c-1}$ when $1<c<n-c,\ \frac{1}{n-c}<\frac{c}{n-c}<1,$ we get

$$\left[1+\frac{2(n-c-1)}{c-1}\right]\ln\left[1+\frac{c-1}{n-c-1}\right]-2=\left[1+\frac{2(n-c-1)}{c-1}\right]\left[\begin{array}{l}\frac{c-1}{n-c-1}-\frac{(c-1)^2}{2(n-c-1)^2}\\+\frac{(c-1)^3}{3(n-c-1)^3}-\frac{(c-1)^4}{4(n-c-1)^4}\\+\frac{(c-1)^5}{5(n-c-1)^5}-......\end{array}\right]-2$$

$$=\frac{c-1}{n-c-1}-\frac{(c-1)^2}{2(n-c-1)^2}+\frac{(c-1)^3}{3(n-c-1)^3}-\frac{(c-1)^4}{4(n-c-1)^4}+\frac{(c-1)^5}{5(n-c-1)^5}-......+2-\frac{c-1}{n-c-1}$$

$$+\frac{2(c-1)^2}{3(n-c-1)^2}-\frac{2(c-1)^3}{4(n-c-1)^3}+\frac{2(c-1)^4}{5(n-c-1)^4}-\frac{2(c-1)^5}{6(n-c-1)^5}+...-2$$

$$=\frac{(c-1)^2}{(n-c-1)^2}\left(\frac{2}{3}-\frac{1}{2}\right)+\frac{(c-1)^3}{(n-c-1)^3}\left(\frac{1}{3}-\frac{1}{2}\right)+\frac{(c-1)^4}{(n-c-1)^4}\left(\frac{2}{5}-\frac{1}{4}\right)+\frac{(c-1)^5}{(n-c-1)^5}\left(\frac{1}{5}-\frac{1}{3}\right)$$

$$=\frac{(c-1)^2}{6(n-c-1)^2}-\frac{(c-1)^3}{6(n-c-1)^3}+\frac{3(c-1)^4}{20(n-c-1)^4}-\frac{2(c-1)^5}{15(n-c-1)^5}+...$$

$$=\frac{(c-1)^2}{6(n-c-1)^2}\left[1-\frac{c-1}{n-c-1}\right]+\frac{(c-1)^4}{60(n-c-1)^4}\left[9-\frac{8(c-1)}{n-c-1}\right]+... \tag{A.3}$$

According to (A.1), (A.2) and (A.3), we have

$$\hat{\beta}_{QEBS\,1}-\hat{\beta}_{QEBS\,3}>0,\quad \hat{\beta}_{QEBS\,2}-\hat{\beta}_{QEBS\,1}>0$$

That is $\hat{\beta}_{QEBS\,3} < \hat{\beta}_{QEBS\,1} < \hat{\beta}_{QEBS\,2}$

(ii) From (A.1) and (A.2), we get

$$\lim_{n\to\infty}\left(\hat{\beta}_{QEBS\,1} - \hat{\beta}_{QEBS\,3}\right) = \left(\frac{D}{c-1}\right)\lim_{n\to\infty}\left\{\begin{array}{l}\frac{(c-1)^2}{6(n-c-1)^2}\left[1-\frac{c-1}{n-c-1}\right]\\ +\frac{(c-1)^4}{60(n-c-1)^4}\left[9-\frac{8(c-1)}{n-c-1}\right]+\dots\end{array}\right\} = 0 \qquad \text{(A.4)}$$

$$\lim_{n\to\infty}\left(\hat{\beta}_{QEBS\,2} - \hat{\beta}_{QEBS\,1}\right) = \left(\frac{D}{c+1}\right)\lim_{n\to\infty}\left\{\begin{array}{l}\frac{(c-1)^2}{6(n-c-1)^2}\left[1-\frac{c-1}{n-c-1}\right]\\ +\frac{(c-1)^4}{60(n-c-1)^4}\left[9-\frac{8(c-1)}{n-c-1}\right]+\dots\end{array}\right\} = 0 \qquad \text{(A.5)}$$

According to (A.4) and (A.5), we have

That is $\lim_{n\to\infty}\hat{\beta}_{QEBS\,1} = \lim_{n\to\infty}\hat{\beta}_{QEBS\,2} = \lim_{n\to\infty}\hat{\beta}_{QEBS\,3}$

Proof of Theorem 10.

(i) From (35), (36) and (37), we get

$$\hat{\beta}_{QEBW\,1} - \hat{\beta}_{QEBW\,3} = \left(\frac{D}{c-1}\right)\left\{\left[1+\frac{2(n-c-2)}{c-1}\right]\ln\left[1+\frac{c-1}{n-c-2}\right]-2\right\} \qquad \text{(A.6)}$$

$$\hat{\beta}_{QEBW\,2} - \hat{\beta}_{QEBW\,1} = \left(\frac{D}{c+1}\right)\left\{\left[1+\frac{2(n-c-2)}{c-1}\right]\ln\left[1+\frac{c-1}{n-c-2}\right]-2\right\} \qquad \text{(A.7)}$$

Based on (A.3), we can obtain

$$\left[1+\frac{2(n-c-2)}{c-1}\right]\ln\left[1+\frac{c-1}{n-c-2}\right]-2 = \frac{(c-1)^2}{6(n-c-2)^2}\left[1-\frac{c-1}{n-c-2}\right] + \frac{(c-1)^4}{60(n-c-2)^4}\left[9-\frac{8(c-1)}{n-c-2}\right]+\dots \qquad \text{(A.8)}$$

According to (A.6), (A.7) and (A.8), we have

$$\hat{\beta}_{QEBW\,1} - \hat{\beta}_{QEBW\,3} > 0, \quad \hat{\beta}_{QEBW\,2} - \hat{\beta}_{QEBW\,1} > 0$$

That is $\hat{\beta}_{QEBW\,3} < \hat{\beta}_{QEBW\,1} < \hat{\beta}_{QEBW\,2}$

(ii) From (A.6) and (A.7), we get

$$\lim_{n\to\infty}\left(\hat{\beta}_{QEBW\,1}-\hat{\beta}_{QEBW\,3}\right)=\left(\frac{D}{c-1}\right)\lim_{n\to\infty}\left\{\begin{array}{l}\frac{(c-1)^2}{6(n-c-2)^2}\left[1-\frac{c-1}{n-c-2}\right]\\+\frac{(c-1)^4}{60(n-c-2)^4}\left[9-\frac{8(c-1)}{n-c-2}\right]+\ldots\end{array}\right\}=0 \tag{A.9}$$

$$\lim_{n\to\infty}\left(\hat{\beta}_{QEBW\,2}-\hat{\beta}_{QEBW\,1}\right)=\left(\frac{D}{c+1}\right)\lim_{n\to\infty}\left\{\begin{array}{l}\frac{(c-1)^2}{6(n-c-2)^2}\left[1-\frac{c-1}{n-c-2}\right]\\+\frac{(c-1)^4}{60(n-c-2)^4}\left[9-\frac{8(c-1)}{n-c-2}\right]+\ldots\end{array}\right\}=0 \tag{A.10}$$

According to (A.9) and (A.10), we have

That is $\lim_{n\to\infty}\hat{\beta}_{QEBW\,1}=\lim_{n\to\infty}\hat{\beta}_{QEBW\,2}=\lim_{n\to\infty}\hat{\beta}_{QEBW\,3}$

Proof of Theorem 11.

(i) From (39), (40) and (41), we get

$$\hat{\beta}_{QEBE1}-\hat{\beta}_{QEBE3}=\left(\frac{D}{c-1}\right)\left\{\left[1+\frac{2(n-c)}{c-1}\right]\ln\left[1+\frac{c-1}{n-c}\right]-2\right\} \tag{A.11}$$

$$\hat{\beta}_{QEBE2}-\hat{\beta}_{QEBE1}=\left(\frac{D}{c+1}\right)\left\{\left[1+\frac{2(n-c)}{c-1}\right]\ln\left[1+\frac{c-1}{n-c}\right]-2\right\} \tag{A.12}$$

Based on (A.3), we can get

$$\left[1+\frac{2(n-c)}{c-1}\right]\ln\left[1+\frac{c-1}{n-c}\right]-2=\frac{(c-1)^2}{6(n-c)^2}\left[1-\frac{c-1}{n-c}\right]+\frac{(c-1)^4}{60(n-c)^4}\left[9-\frac{8(c-1)}{n-c}\right]+.. \tag{A.13}$$

According to (A.11), (A.12 and (A.13), we have

$$\hat{\beta}_{QEBE1}-\hat{\beta}_{QEBE3}>0,\quad \hat{\beta}_{QEBE2}-\hat{\beta}_{QEBE1}>0$$

That is $\hat{\beta}_{QEBE3}<\hat{\beta}_{QEBE1}<\hat{\beta}_{QEBE2}$

(ii) From (A.11), (A.12) and (A.13), we get

$$\lim_{n\to\infty}\left(\hat{\beta}_{QEBE\,1}-\hat{\beta}_{QEBE\,3}\right)=\left(\frac{D}{c-1}\right)\lim_{n\to\infty}\left\{\begin{array}{l}\frac{(c-1)^2}{6(n-c)^2}\left[1-\frac{c-1}{n-c}\right]\\+\frac{(c-1)^4}{60(n-c)^4}\left[9-\frac{8(c-1)}{n-c}\right]+\ldots\end{array}\right\}=0 \tag{A.14}$$

$$\lim_{n\to\infty}\left(\hat{\beta}_{QEBE\,2}-\hat{\beta}_{QEBE\,1}\right)=\left(\frac{D}{c+1}\right)\lim_{n\to\infty}\left\{\begin{array}{l}\frac{(c-1)^2}{6(n-c)^2}\left[1-\frac{c-1}{n-c}\right]\\+\frac{(c-1)^4}{60(n-c)^4}\left[9-\frac{8(c-1)}{n-c}\right]+\ldots\end{array}\right\}=0 \quad (A.15)$$

According to (A.14) and (A.15), we have

That is $\lim_{n\to\infty}\hat{\beta}_{QEBE1}=\lim_{n\to\infty}\hat{\beta}_{QEBE2}=\lim_{n\to\infty}\hat{\beta}_{QEBE3}$

Proof of Theorem 12.

(i) From (43), (44) and (45), we get

$$\hat{\beta}_{QEBM\,1}-\hat{\beta}_{QEBM\,3}=\left(\frac{D}{c-1}\right)\left\{\left[1+\frac{2(n-c+1)}{c-1}\right]\ln\left[1+\frac{c-1}{n-c+1}\right]-2\right\} \quad (A.16)$$

$$\hat{\beta}_{QEBM\,2}-\hat{\beta}_{QEBM\,1}=\left(\frac{D}{c+1}\right)\left\{\left[1+\frac{2(n-c+1)}{c-1}\right]\ln\left[1+\frac{c-1}{n-c+1}\right]-2\right\} \quad (A.17)$$

Based on (A.3), we can obtain

$$\left[1+\frac{2(n-c+1)}{c-1}\right]\ln\left[1+\frac{c-1}{n-c+1}\right]-2=\frac{(c-1)^2}{6(n-c+1)^2}\left[1-\frac{c-1}{n-c+1}\right]+\frac{(c-1)^4}{60(n-c+1)^4}\left[9-\frac{8(c-1)}{n-c+1}\right]+\ldots \quad (A.18)$$

According to (A.16), (A.17) and (A.18), we have

$$\hat{\beta}_{QEBM1}-\hat{\beta}_{QEBM3}>0,\quad \hat{\beta}_{QEBM2}-\hat{\beta}_{QEBM1}>0$$

That is $\hat{\beta}_{QEBM\,3}<\hat{\beta}_{QEBM1}<\hat{\beta}_{QEBM\,2}$

(ii) From (A.16), (A.17) and (A.18), we get

$$\lim_{n\to\infty}\left(\hat{\beta}_{QEBM\,1}-\hat{\beta}_{QEBM\,3}\right)=\left(\frac{D}{c-1}\right)\lim_{n\to\infty}\left\{\begin{array}{l}\frac{(c-1)^2}{6(n-c+1)^2}\left[1-\frac{c-1}{n-c+1}\right]\\+\frac{(c-1)^4}{60(n-c+1)^4}\left[9-\frac{8(c-1)}{n-c+1}\right]+\ldots\end{array}\right\}=0 \quad (A.19)$$

$$\lim_{n\to\infty}\left(\hat{\beta}_{QEBM\,2}-\hat{\beta}_{QEBM\,1}\right)=\left(\frac{D}{c+1}\right)\lim_{n\to\infty}\left\{\begin{array}{l}\frac{(c-1)^2}{6(n-c+1)^2}\left[1-\frac{c-1}{n-c+1}\right]\\+\frac{(c-1)^4}{60(n-c+1)^4}\left[9-\frac{8(c-1)}{n-c+1}\right]+\ldots\end{array}\right\}=0 \quad (A.20)$$

According to (A.19) and (A.20), we have

That is $\lim_{n\to\infty}\hat{\beta}_{QEBM\,1}=\lim_{n\to\infty}\hat{\beta}_{QEBM\,2}=\lim_{n\to\infty}\hat{\beta}_{QEBM\,3}$

APPENDIX - 2

Proof of Theorem 13.

(i) From (47), (48) and (49), we can obtain

$$\hat{\beta}_{EBS1} - \hat{\beta}_{EBS3} == \frac{c-1}{6\alpha H} \tag{B.1}$$

$$\hat{\beta}_{EBS2} - \hat{\beta}_{EBS1} = \frac{(c-1)^2}{6\alpha H\,(c+1)} \tag{B.2}$$

According to (B.1) and (B.2), we have

$$\hat{\beta}_{EBS1} - \hat{\beta}_{EBS3} > 0, \quad \hat{\beta}_{EBS2} - \hat{\beta}_{EBS1} > 0$$

That is $\hat{\beta}_{EBS3} < \hat{\beta}_{EBS1} < \hat{\beta}_{EBS2}$

(ii) From (B.1) and (B.2), we get

$$\lim_{H\to\infty}\left(\hat{\beta}_{EBS1} - \hat{\beta}_{EBS3}\right) = \left(\frac{c-1}{6}\right)\lim_{H\to\infty}\left[\frac{1}{H}\right] = 0 \tag{B.3}$$

$$\lim_{H\to\infty}\left(\hat{\beta}_{EBS2} - \hat{\beta}_{EBS1}\right) = \frac{(c-1)^2}{6(c+1)}\lim_{H\to\infty}\left[\frac{1}{H}\right] = 0 \tag{B.4}$$

According to (B.3) and (B.4), we have

That is $\lim_{H\to\infty}\hat{\beta}_{EBS1} = \lim_{H\to\infty}\hat{\beta}_{EBS2} = \lim_{H\to\infty}\hat{\beta}_{EBS3}$

Proof of Theorem 14.

(i) From (51), (52) and (53), we can obtain

$$\hat{\beta}_{EBW1} - \hat{\beta}_{EBW3} == \frac{c-1}{6\alpha H} \tag{B.5}$$

$$\hat{\beta}_{EBW2} - \hat{\beta}_{EBW1} == \frac{(c-1)^2}{6\alpha H\,(c+1)} \tag{B.6}$$

According to (B.5) and (B.6), we have

$$\hat{\beta}_{EBW1} - \hat{\beta}_{EBW3} > 0, \quad \hat{\beta}_{EBW2} - \hat{\beta}_{EBW1} > 0$$

That is $\hat{\beta}_{EBW3} < \hat{\beta}_{EBW1} < \hat{\beta}_{EBW2}$

(ii) Based on (B.3) and (B.4), we have

That is $\lim_{H\to\infty}\hat{\beta}_{EBW1} = \lim_{H\to\infty}\hat{\beta}_{EBW2} = \lim_{H\to\infty}\hat{\beta}_{EBW3}$

Proof of Theorem 15.

(i) From (55), (56) and (57), we can obtain

$$\hat{\beta}_{EBE1} - \hat{\beta}_{EBE3} == \frac{c-1}{6\alpha H} \tag{B.7}$$

$$\hat{\beta}_{EBE2} - \hat{\beta}_{EBE1} == \frac{(c-1)^2}{6\alpha H(c+1)} \tag{B.8}$$

According to (B.7) and (B.8), we have

$$\hat{\beta}_{EBE1} - \hat{\beta}_{EBE3} > 0, \quad \hat{\beta}_{EBE2} - \hat{\beta}_{EBE1} > 0$$

That is $\hat{\beta}_{EBE3} < \hat{\beta}_{EBE1} < \hat{\beta}_{EBE2}$

(ii) Based on (B.3) and (B.4), we have

That is $\lim_{H\to\infty} \hat{\beta}_{EBE1} = \lim_{H\to\infty} \hat{\beta}_{EBE2} = \lim_{H\to\infty} \hat{\beta}_{EBE3}$

Proof of Theorem 16.

(i) From (59), (60) and (61), we can obtain

$$\hat{\beta}_{EBM1} - \hat{\beta}_{EBM3} == \frac{c-1}{6\alpha H} \tag{B.9}$$

$$\hat{\beta}_{EBM2} - \hat{\beta}_{EBM1} == \frac{(c-1)^2}{6\alpha H\,(c+1)} \tag{B.10}$$

According to (B.9) and (B.10), we have

$$\hat{\beta}_{EBM1} - \hat{\beta}_{EBM3} > 0, \quad \hat{\beta}_{EBM2} - \hat{\beta}_{EBM1} > 0$$

That is $\hat{\beta}_{EBM3} < \hat{\beta}_{EBM1} < \hat{\beta}_{EBM2}$

(ii) Based on (B.3) and (B.4), we have

That is $\lim_{H\to\infty} \hat{\beta}_{EBM1} = \lim_{H\to\infty} \hat{\beta}_{EBM2} = \lim_{H\to\infty} \hat{\beta}_{EBM3}$

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://sciencedomain.org/review-history/16611