British Journal of Mathematics & Computer Science
11(2): 1-10, 2015, Article no.BJMCS.18477
ISSN: 2231-0851

SCIENCEDOMAIN *international*
www.sciencedomain.org

A Note on Common Fixed Point Theorems in Menger Space Using Weak Compatibility

Rajesh Shrivastava[1], Arihant Jain[2*] and Amit Kumar Gupta[1]

[1]Institute for Excellence in Higher Education, Bhopal (M.P.) India.
[2]Department of Applied Mathematics, Shri Guru Sandipani Institute of Technology and Science, Ujjain (M.P.) 456 550, India.

Article Information

DOI: 10.9734/BJMCS/2015/18477
Editor(s):
(1) Dragoş - Pătru Covei, Department of Mathematics, University Constantin Brâncuşi of Târgu-Jiu, România.
Reviewers:
(1) Anonymous, Chandigarh University, India.
(2) Anonymous, Malwa Institute of Technology, Indore, India.
(3) Anonymous, University in Bilaspur, India.
Complete Peer review History: http://sciencedomain.org/review-history/10651

Original Research Article

Received: 24 April 2015
Accepted: 25 July 2015
Published: 23 August 2015

Abstract

In this paper, we prove a unique common fixed point theorem for six weak compatible mappings in Menger space which is an alternate result of Pant et al. [1].

Keywords: Compatible maps; weak compatible maps; Menger space and common fixed point.

AMS subject classification (2000): Primary 47H10, Secondary 54H25.

1 Introduction

Menger [2] introduced the notion of probabilistic metric spaces, which is a generalization of metric spaces. Sehgal and Bharucha-Reid [3] initiated the study of fixed points in probabilistic metric spaces. This study was expanded rapidly with the inspiring works of Schweizer and Sklar [4]. The theory of probabilistic metric spaces is of fundamental importance in probabilistic functional analysis.

The concept of weak compatible mappings was given by Jungck and Rhoades [5] in metric space. The weak commuting concept in metric space has been introduced by Sessa [6]. In 1996, the more generalized concept of this has been given by Jungck [7] in metric space. In 1991, Mishra [8] introduced the notion of compatible mappings in the setting of probabilistic metric space. By using the fact that weak compatible maps are more general than that of compatible maps, Jain et al. [9] proved interesting fixed point theorems.

**Corresponding author: E-mail: arihant2412@gmail.com;*

The present paper deals with the theorem using six self mappings through weak-compatibility, which turns out to be an alternate result of Pant et al. [1]. We support our theorem by providing a suitable example.

2 Preliminaries

Definition 2.1. [8] A real valued function **F** on the set of real numbers is called a *distribution* if it is non-decreasing, left continuous with

$\inf_{u\in R}\mathbf{F}(u)=0$ and $\sup_{u\in R}\mathbf{F}(u) = 1$.

The Heaviside distribution function H is a distribution function defined by

$$H(u) = \begin{cases} 0 & \text{if } u \le 0 \\ 1 & \text{if } u > 0 \end{cases}.$$

Definition 2.2. [8] A *t-norm* is a function $t:[0, 1] \times [0, 1] \to [0, 1]$ if $([0,1], t)$ is an abelian topological monoid with unit 1 such that $t(c, d) \ge t(a, b)$; for $c \ge a$, $d \ge b$,

Definition 2.3. [8] A *probabilistic metric space (PM-space)* is an ordered pair (X, **F**), where X is a set and **F** is function defined on X × X into the set of distribution functions such that if x, y and z are points of X, then

(PM-1) $F_{x,y}(u_1) = 1$, for all $x > 0$, if and only if $x = y$;

(PM-2) $F_{x,y}(0) = 0$;

(PM-3) $F_{x,y} = F_{y,x}$;

(PM-4) If $F_{x,y}(u_1) = 1$ and $F_{y,z}(u_2) = 1$ then $F_{x,z}(u_1 + u_2) = 1$, for all $u_1, u_2 > 0$.

Definition 2.4. [4] A *Menger space* is an ordered triple (X, **F**, t) where (X, **F**) is a PM-space and t is a t-norm satisfying the following condition:

(PM-5) $F_{x,z}(u_1 + u_2) \ge t\{F_{x,y}(u_1), F_{y,z}(u_2)\}$, for all x, y, z in X and $u_1, u_2 \ge 0$.

Definition 2.5. [4] A sequence $\{x_n\}$ in (X, **F**, t) is said to be *converge to a point* $x \in X$ if for every $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer $M(\varepsilon, \lambda)$ such that

$F_{x_n, x}(\varepsilon) > 1 - \lambda$ for all $n \ge M(\varepsilon, \lambda)$.

Further $\{x_n\}$ is said to be *Cauchy sequence* if for every $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer $M(\varepsilon, \lambda)$ such that

$F_{x_n, x_m}(\varepsilon) > 1 - \lambda$ for all $m, n \ge M(\varepsilon, \lambda)$.

A Menger PM-space (X, **F**, t) with continuous t-norm is said to be *complete* if every Cauchy sequence in it converges to a point in it.

Definition 2.6. [9] Suppose (X, **F**, t) be a Menger space. Two self mappings A and S are said to be weak compatible if they commute at their coincidence points i.e.

$Ax = Sx$ for $x \in X$ implies $ASx = SAx$.

Definition 2.7. [10] Suppose (X, **F**, t) be a Menger space. Self mappings A and S are said to be *compatible* if $F_{ASx_n, SAx_n}(x) \to 1$ for all $x > 0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Sx_n \to u$ for some u in X, as $n \to \infty$.

Definition 2.8. [1] Suppose (X, **F**, t) be a Menger space. Self maps S and T are said to be *semi-compatible* if $F_{STx_n, Tu}(x) \to 1$ for all $x > 0$, whenever $\{x_n\}$ is a sequence in X such that $Sx_n, Tx_n \to u$ for some u in X, as $n \to \infty$.

It follows that if (S, T) is semi compatible and Sx = Tx then STx = TSx. Thus if the pair (S, T) is semi-compatible then it is weakly compatible. The converse is not true as seen in [11].

Remark 2.1. [11] Every semi-compatible pair of self-maps is weak compatible but the reverse is not true always.

Lemma 2.1. [12] Suppose (X, **F**, t) be a Menger space with continuous t-norm t with $t(a, a) \geq a$ and $\{x_n\}$ be a sequence. If there exists a constant $k \in (0, 1)$ such that $F_{x_n,x_{n+1}}(kt) \geq F_{x_{n-1}, x_n}(t)$ for all $t \geq 0$ and n = 1, 2, 3, ..., then $\{x_n\}$ is a Cauchy sequence in X.

Lemma 2.2. [12] Suppose in a Menger space (X, **F**, t), there exists a constant $k \in (0, 1)$ with $F_{x, y}(kt) \geq F_{x, y}(t)$ for all $x, y \in X$ and $t > 0$, then x = y.

Proposition 2.1. [8] In a Menger space (X, **F**, t) if $t(x, x) \geq x$, $\forall x \in [0, 1]$ then $t(a, b) = \min\{a, b\}$, $\forall a, b \in [0, 1]$.

Proposition 2.2. Let $\{x_n\}$ be a Cauchy sequence in a Menger space (X, **F**, t) with continuous t-norm t. If the subsequence $\{x_{2n}\}$ converges to x in X, then $\{x_n\}$ also converges to x.

Proof. As $\{x_{2n}\}$ converges to x, we have

$$F_{x_n,x}(\varepsilon) \geq t\left(F_{x_n,x_{2n}}\left(\frac{\varepsilon}{2}\right), F_{x_{2n},x}\left(\frac{\varepsilon}{2}\right)\right).$$

Taking limit as $n \to \infty$ we get $\lim_{n\to\infty} F_{x_n, x}(\varepsilon) \geq t(1, 1)$, which gives $\lim_{n\to\infty} F_{x_n, x}(\varepsilon) = 1$; for all $\varepsilon > 0$ and the result follows.

A class of implicit relation. Let Φ be the set of all real continuous functions $\phi : (R^+)^4 \to R$, non-decreasing in the first argument with the property :

a. For $u, v \geq 0$, $\phi(u, v, v, u) \geq 0$ or $\phi(u,v,u,v) \geq 0$ implies that $u \geq v$.
b. $\phi(u, u, 1, 1) \geq 0$ implies that $u \geq 1$.

Example 2.1. Define $\phi(t_1,t_2,t_3,t_4) = 18t_1 - 16t_2 + 8t_3 - 10t_4$. Then $\phi \in \Phi$.

Noteworthy results related to fixed point theorems using compatibility of type (A) and weak compatibility in Menger space was given in Singh et al. [13,14].

3 Main Results

Theorem 3.1. Let A, B, L, M, S and T be self mappings on a Menger space (X, **F**, t) with continuous t-norm t satisfying:

$L(X) \subseteq ST(X)$, $M(X) \subseteq AB(X)$; (3.1.1)

$AB = BA$, $ST = TS$, $LB = BL$, $MT = TM$; (3.1.2)

One of ST(X), M(X), AB(X) or L(X) is complete; (3.1.3)

The pairs (L, AB) and (M, ST) are weak-compatible; (3.1.4)

for some $\phi \in \Phi$, there exists $k \in (0, 1)$ such that for all $x, y \in X$ and $t > 0$, (3.1.5)

$$\phi(F_{Lx,\, My}(kt), F_{ABx,\, STy}(t), F_{Lx,\, ABx}(t), F_{My,\, STy}(kt)) \geq 0$$

then A, B, L, M, S and T have a unique common fixed point in X.

Proof. Suppose $x_0 \in X$ be any arbitrary point for which there exist $x_1, x_2 \in X$ such that

$$Lx_0 = STx_1 = y_0 \text{ and } Mx_1 = ABx_2 = y_1.$$

Inductively, construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$Lx_{2n} = STx_{2n+1} = y_{2n} \text{ and } Mx_{2n+1} = ABx_{2n+2} = y_{2n+1}$$

for n = 0, 1, 2, ...

Step 1. Using (3.1.5) with $x = x_{2n}$ and $y = x_{2n+1}$, we get

$$\phi(F_{Lx_{2n},\, Mx_{2n+1}}(kt), F_{ABx_{2n},\, STx_{2n+1}}(t), F_{Lx_{2n},\, ABx_{2n}}(t), F_{Mx_{2n+1},\, STx_{2n+1}}(kt)) \geq 0.$$

Letting $n \to \infty$, we get

$$\phi(F_{y_{2n},\, y_{2n+1}}(kt), F_{y_{2n-1},\, y_{2n}}(t), F_{y_{2n},\, y_{2n-1}}(t), F_{y_{2n+1},\, y_{2n}}(kt)) \geq 0.$$

Using (a), we get

$$F_{y_{2n},\, y_{2n+1}}(kt) \geq F_{y_{2n-1},\, y_{2n}}(t).$$

Therefore, for all n even or odd, we have

$$F_{y_n,\, y_{n+1}}(kt) \geq F_{y_{n-1},\, y_n}(t).$$

Therefore, by lemma 2.1, $\{y_n\}$ is a Cauchy sequence in X.

Case I. ST(X) is complete.

In this case $\{y_{2n}\} = \{STx_{2n+1}\}$ is a Cauchy sequence in ST(X), which is complete. Thus $\{y_{2n+1}\}$ converges to some $z \in ST(X)$.

By Proposition 2.2, we have

$$\{Mx_{2n+1}\} \to z \quad \text{and} \quad \{STx_{2n+1}\} \to z$$

$$\{Lx_{2n}\} \to z \quad \text{and} \quad \{ABx_{2n}\} \to z.$$

As $z \in ST(X)$ there exists $u \in X$ such that $z = STu$.

Step I. Using (3.1.5) with $x = x_{2n}$ and $y = u$, we get

$$\phi(F_{Lx_{2n}, Mu}(kt), F_{ABx_{2n}, STu}(t), F_{Lx_{2n}, ABx_{2n}}(t), F_{Mu, STu}(kt)) \geq 0.$$

Taking limit as $n \to \infty$, we get

$$\phi(F_{z, Mu}(kt), F_{z, z}(t), F_{z, z}(t), F_{Mu, z}(kt)) \geq 0$$

$$\phi(F_{z, Mu}(kt), 1, 1, F_{z, Mu}(kt)) \geq 0$$

Using (a) we have $F_{z, Mu}(kt) \geq 1$, for all $t > 0$.

Hence $F_{z, Mu}(t) = 1$.

Thus $z = Mu$.

Hence $STu = Mu = z$. As (M, ST) is weak-compatible so we have $Mz = STz$.

Step II. Using (3.1.5) with $x = x_{2n}$ and $y = z$, we get

$$\phi(F_{Lx_{2n}, Mz}(kt), F_{ABx_{2n}, STz}(t), F_{Lx_{2n}, ABx_{2n}}(t), F_{Mz, STz}(kt)) \geq 0$$

Taking limit as $n \to \infty$, we get

$$\phi(F_{z, Mz}(kt), F_{z, z}(t), F_{z, z}(t), F_{Mz, Mz}(kt)) \geq 0$$

$$\phi(F_{z, Mz}(kt), 1, 1, 1) \geq 0.$$

Using (a) we have

$$F_{z, Mz}(kt) \geq 1, \text{ for all } t > 0.$$

Hence $F_{z, Mz}(t) = 1$.

Thus $z = Mz$.

Step III. Using (3.1.5) with $x = x_{2n}$ and $y = Tz$, we get

$\phi(F_{Lx_{2n}, MTz}(kt), F_{ABx_{2n}, STTz}(t), F_{Lx_{2n}, ABx_{2n}}(t), F_{MTz, STTz}(kt)) \geq 0.$

As MT = TM and ST = TS we have MTz = TMz = Tz and ST(Tz) = T(STz) = Tz. Letting $n \to \infty$, we get

$\phi(F_{z, Tz}(kt), F_{z, Tz}(t), F_{z, z}(t), F_{Tz, Tz}(kt)) \geq 0$

$\phi(F_{z, Tz}(kt), F_{z, Tz}(t), 1, 1) \geq 0$

As ϕ is non-decreasing in the first argument, we have

$\phi(F_{z, Tz}(t), F_{z, Tz}(t), 1, 1) \geq 0.$

Using (b), we get

$F_{z, Tz}(t) \geq 1$ for all $t > 0$.

Hence,

$F_{z, Tz}(t) = 1$, for all $t > 0$,

i.e. z = Tz.

Now STz = Tz = z implies Sz = z. Hence Sz = Tz = Mz = z.

Step IV. As $M(X) \subseteq AB(X)$ there exists $v \in X$ such that z = Mz = ABv.

Using (3.1.5) with x = v and $y = x_{2n+1}$, we get

$\phi(F_{Lv, Mx_{2n+1}}(kt), F_{ABv, STx_{2n+1}}(t), F_{Lv, ABv}(t), F_{Mx_{2n+1}, STx_{2n+1}}(kt)) \geq 0.$

Letting $n \to \infty$, we get

$\phi(F_{Lv, z}(kt), F_{z, z}(t), F_{Lv, z}(t), F_{z, z}(kt)) \geq 0$

$\phi(F_{Lv, z}(kt), 1, F_{Lv, z}(t), 1) \geq 0.$

As ϕ is non-decreasing in the first argument, we have

$\phi(F_{Lv, z}(t), 1, F_{Lv, z}(t), 1) \geq 0.$

Using (a), we have

$F_{Lv, z}(t) \geq 1$, for all $t > 0$

which gives Lv = z.

Therefore, ABz = Lz.

Step V. Using (3.1.5) with x = z and $y = x_{2n+1}$, we get

$$\phi(F_{Lz,\, Mx_{2n+1}}(kt), F_{ABz,\, STx_{2n+1}}(t), F_{Lz,\, ABz}(t), F_{Mx_{2n+1},\, STx_{2n+1}}(kt)) \geq 0.$$

Letting n $\to \infty$, we get

$$\phi(F_{Lz,\, z}(kt), F_{Lz,\, z}(t), F_{Lz,\, Lz}(t), F_{z,\, z}(kt)) \geq 0$$

$$\phi(F_{Lz,\, z}(kt), F_{Lz,\, z}(t), 1, 1) \geq 0.$$

As ϕ is non-decreasing in the first argument, we have

$$\phi(F_{Lz,\, z}(t), F_{Lz,\, z}(t), 1, 1) \geq 0.$$

Using (b), we have

$$F_{Lz,\, z}(t) \geq 1, \text{ for all } t > 0$$

which gives Lz = z.

Therefore, ABz = Lz = z.

Step VI. Using (3.1.5) with x = Bz and y = x_{2n+1}, we get

$$\phi(F_{LBz,\, Mx_{2n+1}}(kt), F_{ABBz,\, STx_{2n+1}}(t), F_{LBz,\, ABBz}(t), F_{Mx_{2n+1},\, STx_{2n+1}}(kt)) \geq 0.$$

As BL = LB, AB = BA, so we have L(Bz) = B(Lz) = Bz and AB(Bz) =B(ABz) = Bz. Letting n $\to \infty$, we get

$$\phi(F_{Bz,\, z}(kt), F_{Bz,\, z}(t), F_{Bz,\, Bz}(t), F_{z,\, z}(kt)) \geq 0$$

$$\phi(F_{Bz,\, z}(kt), F_{Bz,\, z}(t), 1, 1) \geq 0.$$

As ϕ is non-decreasing in the first argument, we have

$$\phi(F_{Bz,\, z}(t), F_{Bz,\, z}(t), 1, 1) \geq 0.$$

Using (b), we have

$$F_{Bz,\, z}(t) \geq 1, \text{ for all } t > 0$$

which gives Bz = z and ABz = z implies Az = z.

Therefore Az = Bz = Lz = z.

Combining the results from different steps, we have

Az = Bz = Lz = Mz = Tz = Sz = z.

Hence the six self maps have a common fixed point in this case. Case when L(X) is complete follows from above case as $L(X) \subseteq ST(X)$.

Case II. AB(X) is complete. This case follows by symmetry. As M(X) $\subseteq$ AB(X), therefore the result also holds when M(X) is complete.

Uniqueness. Let u be another common fixed point of A, B, L, M, S and T, then

$$Au = Bu = Lu = Su = Tu = Mu = u.$$

Using (3.1.5) with x = z and y = u, we get

$$\phi(F_{Lz,\, Mu}(kt), F_{ABz,\, STu}(t), F_{Lz,\, ABz}(t), F_{Mu,\, STu}(kt)) \geq 0$$

$$\phi(F_{z,\, u}(kt), F_{z,\, u}(t), F_{z,\, z}(t), F_{u,\, u}(kt)) \geq 0$$

$$\phi(F_{z,\, u}(kt), F_{z,\, u}(t), 1, 1) \geq 0.$$

As ϕ is non-decreasing in the first argument, we have

$$\phi(F_{z,\, u}(t), F_{z,\, u}(t), 1, 1) \geq 0.$$

Using (b), we have

$$F_{z,\, u}(t) \geq 1, \text{ for all } t > 0.$$

Thus, $F_{z,\, u}(t) = 1$,

i.e., z = u.

Therefore, z is a unique common fixed point of A, B, L, M, S and T.

This completes the proof.

If we take B = T = I, the identity map in theorem 3.1, we get the following corollary.

Corollary 3.1. Let A, L, M and S be self mappings on a Menger space (X, **F**, t) with continuous t-norm t satisfying :

L(X) $\subseteq$ S(X), M(X) $\subseteq$ A(X); (3.1.6)

One of S(X), M(X), A(X) or L(X) is complete; (3.1.7)

The pairs (L, A) and (M, S) are weak-compatible; (3.1.8)

for some $\phi \in \Phi$, there exists k $\in$ (0, 1) such that for all x, y $\in$ X and t > 0, (3.1.9)

$$\phi(F_{Lx,\, My}(kt), F_{Ax,\, Sy}(t), F_{Lx,\, Ax}(t), F_{My,\, Sy}(kt)) \geq 0$$

then A, L, M and S have a unique common fixed point in X.

Example 3.1. Let (X, d) be a metric space where X = [0, 1] and (X, **F**, t) be the induced Menger space with $F_{p,q}(\varepsilon) = H(\varepsilon - d(p, q))$, for all p, q $\in$ X and $\varepsilon > 0$. Define self maps L, M, A and S as follows:

$$L(x) = M(x) = \begin{cases} 0, & x \in \left[0, \frac{4}{5}\right] \\ 1 - x, & \text{otherwise} \end{cases}$$

$$A(x) = \begin{cases} 0, & x \in \left[0, \frac{3}{4}\right] \\ 1 - x, & \text{otherwise} \end{cases}$$

$$S(x) = \begin{cases} 0, & x \in \left[0, \frac{1}{3}\right] \\ 1 - x, & \text{otherwise.} \end{cases}$$

Then L(X) = M(X) = $\left[0, \frac{1}{5}\right)$, A(X) = $\left[0, \frac{1}{4}\right]$ and S(X) = $\left[0, \frac{2}{3}\right)$.

Hence, the containment condition (3.1.6) is satisfied. Also, the pairs (L, A) and (M, S) are weak-compatible and A(X) is complete. Further, for k = $\frac{1}{3}$ the condition (3.1.9) is satisfied. Thus, all the conditions of Corollary 3.1 are satisfied and 0 is the unique common fixed point of the mappings A, L, M and S.

4 Conclusion

In view of proposition 2.1, t(a, b) = min{a, b}, theorem 3.1 is an alternate result of Pant et al. [1], reducing the semi-compatibility of the pair (L, AB) to its weak compatibility and dropping the condition of continuity in a Menger space with continuous t-norm.

Acknowledgement

Authors are thankful to the referee for the preparation of this manuscript.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Pant BD, Chauhan S. Common fixed point theorems for semi-compatible mappings using implicit relation. Int. Journal of Math. Analysis. 2009;3(28):1389-1398.

[2] Menger K. Statistical metrics. Proc. Nat. Acad. Sci. USA. 1942;28:535-537.

[3] Sehgal VM. Bharucha-Reid AT. Fixed points of contraction maps on probabilistic metric spaces, Math. System Theory. 1972;6:97-102.

[4] Schweizer B, Sklar A. Statistical metric spaces. Pacific J. Math. 1960;10:313-334.

[5] Jungck G, Rhoades BE. Fixed points for set valued functions without continuity. Indian J. Pure Appl. Math. 1998;29:227-238.

[6] Sessa S. On a weak commutativity condition of mappings in fixed point consideration. Publ. Inst. Math. Beograd. 1982;32(46):146-153.

[7] Jungck G. Compatible mappings and common fixed points. Internat. J. Math. and Math. Sci. 1986; 9(4):771-779.

[8] Mishra SN. Common fixed points of compatible mappings in PM-spaces, Math. Japon. 1991;36(2): 283-289.

[9] Jain A, Chaudhary B. On common fixed point theorems for semi-compatible and occasionally weakly compatible mappings in Menger space, International Journal of Research and Reviews in Applied Sciences. 2013;14(3):662-670.

[10] Jain A, Singh B. Common fixed point theorem in menger space through compatible maps of type (A). Chh. J. Sci. Tech. 2005;2:1-12.

[11] Chandel RS, Verma R. Fixed point theorem in menger space using weakly compatible. Int. J. Pure Appl. Sci. Technol. 2011;7(2):141-148.

[12] Singh SL, Pant BD. Common fixed point theorems in probabilistic metric spaces and extension to uniform spaces. Honam. Math. J. 1984;6:1-12.

[13] Singh B, Jain A, Shah JA. Fixed point theorem using compatibility of type (A) and weak compatibility in Menger space. IJRRAS. 2012;10(3):451-459.

[14] Singh M, Sharma RK, Jain A. Compatible mappings of type (A) and common fixed points in Menger space. Vikram Math. J. 2000;20:68-78.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://sciencedomain.org/review-history/10651