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Deduction of Double Wronskian Solution and Rational-like Solutions for Generalized Non-autonomous Schrodinger Equation

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Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

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ABSTRACT

In this paper, we have presented the bilinear form, a generalized double Wronskian solution of a non-autonomous Schrödinger equation. Furthermore, we found rational-like solutions by taking special case in general solutions.

Keywords: Double wronskian solution; rational-like solution; non-autonomous schrödinger equation; 2010 MSC: 35C15.

1. INTRODUCTION

In the process of searching for explicit exact solutions, various methods have been developed to the non- linear evolution equations (NLEEs), such as the inverse scattering transform [1,2], Bäcklund and Darboux transformations [3,4], Hirota bilinear method [5], the Wronskian technique [6], Lie symmetry method [7] and so on. Among them, the Wronskian technique provides us with a powerful tool to construct exact solutions for many NLEEs. Once we construct the entries of the determination of Wronskian, we are able to give direct and simple verificatio- ns of the solutions. What's more, one can utilize Wronskian technique to find rational solutions, positons, negatons, complexitons and interaction solutions for the integrable equations in Wronskian form [6,10,16,17].

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As we know, the Wronskian technique has been proved very effective in obtaining explicit solutions to classic integrable equations, such as KdV, mKdV, KP, Boussinesq, nonlinear Schrödinger (NLS), derivative NLS equations etc [7-15]. Recently, some authors have begun to use this technique to other non-isospect-ral equations and variable-coefficient models. Sun et al. [18] have researched on double Wronskian solution of the non-isospectral AKNS equation; He et al. [5] have considered on the double Wronskian solution of a non-isospectral KP equation; Lv et al. [10] have presented on Wronskian solution of a generalized variable-coefficient nonlinear Schrödinger equation; Tian et al. [19] have discussed on the double Wronskian solitons and rogue waves for the inhomogeneous nonlinear Schrödinger equation in an inhomogeneous plasma. Pishkoo A et al. [20] have researched on G-Function solutions for Schrödinger equation in cylindrical coordinates system.

In this paper, we would like to apply the Wronskian technique to a generalized nonautonomous NLS equation [21,22]

$$iQ_t + Q_{xx} - 2e^{\lambda t} |Q|^2 Q + \frac{1}{4}\lambda^2 x^2 Q = 0 \tag{1}$$

its Lax pair is as follows [21,22,23]:

$$\phi_x = M\phi, \quad \phi_t = N\phi, \tag{2}$$

where

$$M = i\begin{pmatrix} -\Lambda(t) & q(x,t) \\ -q^*(x,t) & \Lambda(t) \end{pmatrix}, \tag{3a}$$

$$N = i\begin{pmatrix} -|q|^2 - \lambda x\Lambda - 2\Lambda^2 & q_x - i\lambda xq - 2iq\Lambda \\ q_x^* + i\lambda xq^* + 2iq^*\Lambda & |q|^2 + \lambda x\Lambda + 2\Lambda^2 \end{pmatrix}, \tag{3b}$$

and $q(x,t) = e^{\frac{\lambda}{2}t + i\frac{\lambda}{4}x^2} Q, \Lambda_j(t)$ satisfies $\Lambda_{j,t} = \lambda\Lambda_j$. Moreover, equation (1) can be reduced to the following equation:

$$iu_t + u_{xx} - 2e^{2\lambda t} |u|^2 u - i\lambda x u_x = 0, \tag{4}$$

by the transformation

$$Q = e^{\frac{\lambda}{2}t - i\frac{\lambda}{4}x^2} u, \tag{5}$$

so we only consider the equation above.

This paper is organized as follows. In Section 2, based on the Lax pair (3), the deduction of double Wronskian solution of (1) are obtained. In Section 3, rational-like solutions of (1) are presented by taking special case in general solutions. Section 4 is devoted to conclusions.

2. DEDUCTION OF DOUBLE WRONSKIAN SOLUTION OF (1)

In this section, we derive the bilinear form of (4) and obtain the deduction of double Wronskian solution of (1).Through the dependent variable transformation

$$u = \frac{G}{F}, \tag{6}$$

where *F* and *G* are both complex, equation(4) is transformed into the following equations:

$$(iD_t + D_x^2 - i\lambda x D_x) G \cdot F = 0, \tag{7a}$$

$$F^* D_x^2 F \cdot F = -2e^{2\lambda t} FGG^*, \tag{7b}$$

where *D* is well-known Hirota bilinear operator defned by

$$D_t^m D_x^n f \cdot g = (\partial_t - \partial_{t'})^m (\partial_x - \partial_{x'})^n f(t,x) g(t',x')|_{t'=t, x'=x}.$$

Let us observe the matrix equations

$$\phi_x = -iA\phi, \quad \phi_t = (-iA_t x - 2iA^2)\phi, \tag{8a}$$

$$\psi_x = iA\psi, \quad \psi_t = (iA_t x + 2iA^2)\psi, \tag{8b}$$

where

$$\phi = (\phi_1, \phi_2, \cdots, \phi_{2N})^T, \ \psi = (\psi_1, \psi_2, \cdots, \psi_{2N})^T, \tag{9}$$

and $A = (a_{ij})$ is an $2N \times 2N$ arbitrary function matrix of independent of *x* satisfying

$$A_t = \lambda A \tag{10}$$

To use the Wronskian technique, we adopt the compact notation introduced by Freeman and Nimmo [10].

$$W^{N,M}(\phi,\psi) = \det(\phi, \partial_x\phi, \cdots \partial_x^{N-1}\phi; \psi, \partial_x\psi, \cdots \partial_x^{M-1}\psi) = |\widehat{N-1}; \widehat{M-1}|, \tag{11}$$

where $\phi = (\phi_1, \phi_2, \cdots, \phi_{N+M})^T$ and $\psi = (\psi_1, \psi_2, \cdots, \psi_{N+M})^T$. Define

$$F = |\widehat{N-1}; \widehat{N-1}|, \tag{12a}$$

$$\widetilde{G} = 2|\widehat{N}; \widehat{N-2}|, \tag{12b}$$

Where ϕ_j and ψ_j satisfy the conditions (8). Let

$$G = ie^{-\lambda t}\widetilde{G}, \tag{13}$$

then we can deduce

$$\widetilde{G^*} = 2|\widehat{N-2}; \widehat{N}|,$$

$$F^* = (-1)^N F, \quad G^* = (-1)^N 2ie^{-\lambda t} |\widehat{N-2}; \widehat{N}|. \tag{14}$$

Noting

$$G_0 = 2ie^{-\lambda t} \,|\, \widehat{N-2}; \widehat{N} \,|. \tag{15}$$

equation (7) can be transformed into the following equations:

$$(iD_t + D_x^2 - i\lambda x D_x) G \cdot F = 0\,, \tag{16a}$$

$$D_x^2 F \cdot F = -2e^{2\lambda t} G G_0. \tag{16b}$$

In what follows we prove that (16) has the double Wronskian solution

$$F = W^{N,N}(\phi;\psi), \tag{17a}$$

$$G = 2ie^{-\lambda t} W^{N+1,N-1}(\phi;\psi), \tag{17b}$$

$$G_0 = 2ie^{-\lambda t} W^{N-1,N+1}(\phi;\psi). \tag{17c}$$

For convenience of proof, we first give the following required lemmas which their proofs can be seen in Chen et al. [2]:

Lemma 2.1

$$|D,a,b||D,c,d| - |D,a,c||D,b,d| + |D,a,d||D,b,c| = 0\,, \tag{18}$$

where D is an $N \times (N-2)$ matrix and a, b, c, d represent N column vectors.

Lemma 2.2

$$\sum_{j=1}^{N} |\alpha_1, \cdots \gamma\alpha_j, \alpha_{j+1}, \cdots \alpha_N| = \sum_{j=1}^{N} \gamma_j \, |\alpha_1, \cdots \alpha_N|, \tag{19}$$

where α_j $(1 \le j \le N)$ are N -dimensional column vectors

and $\gamma\alpha_j$ denotes $\gamma\alpha_j = (\gamma_1\alpha_{1j}, \gamma_2\alpha_{2j}, \cdots \gamma_N \alpha_{Nj})^T$.

Lemma 2.3 Assume that $P = (p_{ij})$ is an $l \times l$ operator matrix and its entries p_{ij} are differential operators.

$B = (b_{ij})$ is an $l \times l$ function matrix with column vector set b_{ij} and row vector set

b'_j $(i = 1, 2, \cdots l;\ j = 1, 2, \cdots l)$,

then

$$\sum_{i=1}^{l} |b_1,\cdots p_i b_i,\cdots b_l| = \sum_{j=1}^{l} \begin{vmatrix} b'_1 \\ \vdots \\ p'_j b'_j \\ \vdots \\ b'_t \end{vmatrix}, \tag{20}$$

where

$$p_i b_i = (p_{1i} b_{1i}, p_{2i} b_{2i}, \cdots p_{li} b_{li})^T,$$
$$p'_j b'_j = (p_{j1} b_{j1}, p_{j2} b_{j2}, \cdots p_{jl} b_{jl})^T.$$

Lemma 2.4 Suppose $A = (a_{ij})$ is an $2N \times 2N$ arbitrary function matrix of independent of *x* satisfying $A_t = \lambda A$, then, under the condition (8), we have

$$\begin{aligned} &tr(A)|\widehat{N-1};\widehat{N-1}| \\ &= -|\widehat{N-2}, N;\widehat{N-1}| + |\widehat{N-1};\widehat{N-2}, N|, \end{aligned} \tag{21a}$$

$$\begin{aligned} &[tr(A)]^2 |\widehat{N-1};\widehat{N-1}| \\ &= |\widehat{N-3}, N-1, N;\widehat{N-1}| + |\widehat{N-2}, N+1;\widehat{N-1}|, \\ &-2|\widehat{N-2}, N;\widehat{N-2}, N| + |\widehat{N-1};\widehat{N-3}, N-1, N| \\ &+ |\widehat{N-1};\widehat{N-2}, N+1|, \end{aligned} \tag{21b}$$

$$\begin{aligned} &tr(A)|\widehat{N};\widehat{N-2}| \\ &= -|\widehat{N-1}, N+1;\widehat{N-2}| + |\widehat{N};\widehat{N-3}, N-1|, \end{aligned} \tag{21c}$$

$$\begin{aligned} &[tr(A)]^2 |\widehat{N};\widehat{N-2}| \\ &= |\widehat{N-2}, N, N+1;\widehat{N-2}| + |\widehat{N-1}, N+2;\widehat{N-2}| \\ &-2|\widehat{N-1}, N+1;\widehat{N-3}, N-1| + |\widehat{N};\widehat{N-3}, N| \\ &+ |\widehat{N};\widehat{N-4}, N-2, N-1|, \end{aligned} \tag{21d}$$

in general, we obtain

$$\begin{aligned} &\left\{(trA)^2 |\widehat{N-1};\widehat{N-1}|\right\} |\widehat{N};\widehat{N-2}| \\ &= \left\{(trA)|\widehat{N};\widehat{N-2}|\right\}\left\{(trA)|\widehat{N-1};\widehat{N-1}|\right\} \end{aligned}$$

$$=\left\{(t\,rA)^2\,|\,\widehat{N};\widehat{N-2}\,|\right\}|\,\widehat{N-1};\widehat{N-1}\,| \tag{21e}$$

$$\left\{(t\,rA)^2\,|\,\widehat{N-1};\widehat{N-1}\,|\right\}|\,\widehat{N-1};\widehat{N-1}\,|$$
$$=\left\{(t\,rA)\,|\,\widehat{N-1};\widehat{N-1}\,|\right\}\left\{(t\,rA)\,|\,\widehat{N-1};\widehat{N-1}\,|\right\}. \tag{21f}$$

The proofs of lemmas mentioned above can be seen in Chen et al. [**2**].

Theorem 2.1 Equation (16) has double Wronskian solution (17), where ϕ_j ,ψ_j are satisfied by (8). Thus, with the transformation (5), the corresponding double Wronskain solution of (1) can be expressed as

$$Q=\frac{2iW^{N+1,N-1}(\phi;\psi)}{W^{N,N}(\phi;\psi)}e^{-\frac{\lambda}{2}t-i\frac{\lambda}{4}x^2}. \tag{22}$$

Proof. Note $\Delta=2ie^{-\lambda t}$, the derivatives of *F, G* can be easily computed

$$F_x=|\,\widehat{N-2},N;\widehat{N-2}\,|+|\,\widehat{N-1};\widehat{N-2},N\,|, \tag{23a}$$

$$F_{xx}=|\,\widehat{N-3},N-1,N;\widehat{N-1}\,|+|\,\widehat{N-2},N+1;\widehat{N-1}\,|$$
$$+2\,|\,\widehat{N-2},N;\widehat{N-2},N\,|+|\,\widehat{N-1};\widehat{N-3},N-1,N\,|$$
$$+|\,\widehat{N-1};\widehat{N-2},N+1\,|, \tag{23b}$$

$$G_x=\Delta(|\,\widehat{N-1},N+1;\widehat{N-2}\,|+|\,\widehat{N};\widehat{N-3},N-3\,|), \tag{23c}$$

$$G_{xx}=\Delta(|\,\widehat{N-2},N,N+1;\widehat{N-2}\,|$$
$$+|\,\widehat{N-1},N+2;\widehat{N-2}\,|+|\,\widehat{N};\widehat{N-3},N\,|$$
$$+2\,|\,\widehat{N-1},N+1;\widehat{N-3},N-1\,|$$
$$+|\,\widehat{N};\widehat{N-4},N-2,N-1\,|). \tag{23d}$$

From the condition (8), we have

$$F_t=N(N-1)\lambda F$$
$$+\lambda x(|\,\widehat{N-2},N;\widehat{N-1}\,|+|\,\widehat{N-1};\widehat{N-2},N\,|)$$
$$+2i(|\,\widehat{N-1};\widehat{N-3},N-1,N\,|$$

$$-|\widehat{N-3}, N-1, N; \widehat{N-1}| + |\widehat{N-2}, N; \widehat{N-1}|$$
$$-|\widehat{N-1}; \widehat{N-2}, N+1|), \tag{24a}$$

$$G_t = [-2i\alpha x + (N^2 - N + 1)\lambda]\widetilde{G}$$
$$+\lambda x(|\widehat{N-1}, N+1; \widehat{N-2}| + |\widehat{N}; \widehat{N-3}, N-1|)$$
$$-\lambda\widetilde{G} + 2i(|\widehat{N}; \widehat{N-4}, N-2, N-1|$$
$$-|\widehat{N-2}, N, N+1; \widehat{N-2}| + |\widehat{N-1}, N+2; \widehat{N-2}|$$
$$-|\widehat{N}; \widehat{N-3}, N|). \tag{24b}$$

Substituting (23) and (24) into the left-hand side of (16a) and making use of (21e), we get

$$i(G_tF - GF_t) + (G_{xx}F - 2G_xF_x + GF_{xx}) - i\lambda x(G_xF - GF_x)$$
$$= \Delta(|\widehat{N-1}; \widehat{N-1}||\widehat{N-2}, N, N+1; \widehat{N-2}|$$
$$+|\widehat{N-1}; \widehat{N-1}||\widehat{N}; \widehat{N-3}, N|$$
$$+|\widehat{N}; \widehat{N-2}||\widehat{N-2}, N+1; \widehat{N-1}|$$
$$+|\widehat{N}; \widehat{N-2}||\widehat{N-1}; \widehat{N-3}, N-1, N|$$
$$+|\widehat{N-1}, N+1; \widehat{N-2}||\widehat{N-2}, N; \widehat{N-1}|$$
$$+|\widehat{N}; \widehat{N-3}, N-1||\widehat{N-1}; \widehat{N-2}, N|). \tag{25}$$

According to Lemma 2.1, (25) is equal to zero. So the proof of (16a) is finished. Similarly, we have

$$2(F_{xx}F - F_x^2) + 2e^{2\lambda t}GG_0$$
$$= 2\,[|\widehat{N-1}; \widehat{N-1}|(|\widehat{N-3}, N-1, N; \widehat{N-1}|$$
$$+|\widehat{N-2}, N+1; \widehat{N-1}| + 2|\widehat{N-2}, N; \widehat{N-2}, N|$$
$$+|\widehat{N-1}; \widehat{N-3}, N-1, N| + |\widehat{N-1}; \widehat{N-2}, N+1|)$$
$$-|\widehat{N-2}, N; \widehat{N-1}| + |\widehat{N-1}; \widehat{N-2}, N|)^2]$$
$$-8|\widehat{N}; \widehat{N-2}||\widehat{N-2}; N|. \tag{26}$$

Utilizing the identities (21f), the right-hand side of (26) is reduced as

$$8\,[|\widehat{N-1}; \widehat{N-1}||\widehat{N-2}, N; \widehat{N-2}, N| - [|\widehat{N-1}; \widehat{N-2}, N||\widehat{N-2}, N; \widehat{N-1}|$$
$$-[|\widehat{N}; \widehat{N-2}||\widehat{N-2}; \widehat{N}|). \tag{27}$$

Using Lemma 2.1, (27) is equal to zero. Thus we have proved (16b).Therefore, with the transformation (5), the corresponding double Wronskain solution of (1) can be expressed as (22).

3. RATIONAL-LIKE SOLUTIONS OF (1)

In this section, rational-like solutions, which can be seen as the rational function of *x* after discarding the exponential part, are presented for (1) in the generalized double Wronskian form (22).

From (8), we get the general solution

$$\phi = \exp(-iAx - 2i\int_0^i A^2 dt)C, \tag{28a}$$

$$\psi = \exp(iAx + 2i\int_0^i A^2 dt)D, \tag{28b}$$

where

$$C = (c_1, c_2, \cdots c_{2N})^T, \quad D = (d_1, d_2 \cdots d_{2N})^T$$

are constant vectors. Substituting $A = e^{\lambda t} A_0$ into (28) and expanding it leads

$$\phi = \sum_{s=0}^{\infty} \sum_{l=0}^{[\frac{s}{2}]} \frac{(-ia_1 x)^{s-2l}}{(s-2l)!} \frac{(-ia_2)^l}{l!} A_0^s C, \tag{29a}$$

$$\psi = \sum_{s=0}^{\infty} \sum_{l=0}^{[\frac{s}{2}]} \frac{(ia_1 x)^{s-2l}}{(s-2l)!} \frac{(ia_2)^l}{l!} A_0^s D, \tag{29b}$$

where $a_1 = e^{\lambda t}, a_2 = \dfrac{e^{2\lambda t} - 1}{\lambda}$.

If

$$A_0 = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{pmatrix}_{2N \times 2N}, \tag{30}$$

it is obvious to know that $A^{2N} = 0$. Therefore, (29) can be truncated as

$$\phi = \sum_{s=0}^{2N-1} \sum_{l=0}^{[\frac{s}{2}]} \frac{(-ia_1 x)^{s-2l}}{(s-2l)!} \frac{(-ia_2)^l}{l!} A_0^s C, \tag{31a}$$

$$\psi = \sum_{s=0}^{2N-1} \sum_{l=0}^{[\frac{s}{2}]} \frac{(ia_1 x)^{s-2l}}{(s-2l)!} \frac{(ia_2)^l}{l!} A_0^s D. \tag{31b}$$

The components of ϕ and ψ are

$$\phi_j = [c_j + (-ia_1 x)c_{j-1} + (-\frac{a_1 x^2}{2} - ia_2)c_{j-2}$$

$$+ \cdots + c_1 \sum_{l=0}^{[\frac{j-1}{2}]} (-i)^{j-1-l} \frac{(a_1 x)^{j-1-2l}}{(j-1-2l)!} \frac{(a_2)^l}{l!}], \quad (32a)$$

$$\psi_j = [d_j + (ia_1 x)d_{j-1} + (-\frac{a_1 x^2}{2} + ia_2)d_{j-2}$$

$$+ \cdots + d_1 \sum_{l=0}^{[\frac{j-1}{2}]} (i)^{j-1-l} \frac{(a_1 x)^{j-1-2l}}{(j-1-2l)!} \frac{(a_2)^l}{l!}]. \quad (j = 1, 2, \cdots 2N). \quad (32b)$$

Taking $c_1 = d_1 = 1,\ c_k = d_k = 0\ (k = 2, 3, \cdots, 2N)$, then (32) becomes

$$\phi_j = \sum_{l=0}^{[\frac{j-1}{2}]} (-i)^{j-1-l} \frac{(a_1 x)^{j-1-2l}}{(j-1-2l)!} \frac{(a_2)^l}{l!}, \quad (33a)$$

$$\psi_j = \sum_{l=0}^{[\frac{j-1}{2}]} (i)^{j-1-l} \frac{(a_1 x)^{j-1-2l}}{(j-1-2l)!} \frac{(a_2)^l}{l!}. \quad (33b)$$

Thus, we obtain the rational-like solutions, which can be seen as the rational function of *x* after discarding the exponential part, with Wronskian form of (1) from (22).The first three of lower order are

$$Q = -\frac{i}{x} e^{-\frac{\lambda t}{2} - i\frac{\lambda x^2}{4}}, \quad (34a)$$

$$Q = \frac{2\lambda x e^{\lambda t}(-3 + 3e^{2\lambda t} + i\lambda x^2 e^{2\lambda t})}{-3 + 6e^{2\lambda t} - 3e^{4\lambda t} + \lambda^2 x^4 e^{4\lambda t}} e^{-\frac{\lambda t}{2} - i\frac{\lambda x^2}{4}}, \quad (34b)$$

$$Q = \frac{Q_1 + Q_2 + Q_3}{Q_4}, \quad (34c)$$

where

$$Q_1 = 24\lambda^3 x^6 e^{6\lambda t}(1 - e^{2\lambda t}) e^{-\frac{\lambda t}{2} - i\frac{\lambda x^2}{4}},$$

$$Q_2 = 135i(1-4e^{2\lambda t}+6e^{4\lambda t}+e^{8\lambda t}-4e^{6\lambda t})e^{-\frac{\lambda t}{2}-i\frac{\lambda x^2}{4}},$$

$$Q_3 = 3i\lambda^2 x^4 e^{4\lambda t}(-30+\lambda^2 x^4-30\lambda^2 x^4 e^{8\lambda t}+60e^{2\lambda t})e^{-\frac{\lambda t}{2}-i\frac{\lambda x^2}{4}},$$

$$Q_4 = x(\lambda^4 x^8 e^{8\lambda t}-18\lambda^2 x^4 e^{8\lambda t}-18\lambda^2 x^4 e^{4\lambda t}$$
$$+36\lambda^2 x^4 e^{6\lambda t}-135e^{8\lambda t}+540e^{6\lambda t}-810e^{4\lambda t}+540e^{2\lambda t}-135).$$

These solutions can be verified by direct substitution into (1). If we set ¸ $\lambda \to 0$, the solutions (34) turn out to be

$$Q = -\frac{i}{x}, \quad (35a)$$

$$Q = \frac{12xt+2ix^3}{x^4-12t^2}, \quad (35b)$$

$$Q = \frac{-48x^6t-3ix^8+360ix^4t-i2160t^4}{x^9-72x^5t^2-2160xt^4}, \quad (35c)$$

which are the rational solutions to the standard NLS equation (dark case).

4. CONCLUSION

In summary, we have given the double Wronskian solutions which satisfy matrix equation of a non-autonomous Schrödinger equation through the Hirota method and the Wronskian technique. Moreover, rational-- like solutions of the equation are obtained by taking special case in general solutions. It is believed that the methods used in the paper may be applied for some integrable non-autonomous models and several other variable-coefficient NLEEs. An open problem is whether there exist the multisoliton solutions in the Wronskian form for dark case of NLS-type equations, what the conditions if they exist and how to construct the entries in the Wronskian determinant. We expect to discuss this problem elsewhere.

COMPETING INTERESTS

Authors declare that there are no competing interests.

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