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Paracontact Finsler Structures on Vector Bundles

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Abstract

In this paper, we study the almost paracontact, almost paracontact metric, paracontact metric, K-paracontact and para-Sasakian Finsler structures on vector bundles and give some characterizations for these geometric structures. Also, the curvature of a paracontact Finsler manifold is given and some results for Ricci semi-symmetric para-Sasakian Finsler manifolds and para-Sasakian Finsler manifolds with η-parallel Ricci tensor are obtained with the aid of Ricci tensor and scalar curvature of Finsler structure.

Keywords: Vector Bundle; Vertical Distribution; Horizontal Distribution; Finsler Connection; Nijenhuis Tensor Field

2010 Mathematics Subject Classification: 53C15; 53C25; 53C60; 58B20

1 Introduction

In [1], R. Miron has interested in the differential geometry of vector bundles and used the Finsler geometry to simplify the theory. For this, he has seen that the Finsler connection on the total space

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E of a vector bundle $\xi = (E, \pi, M)$ is fundamental. To define it, firstly he has defined the notion of the nonlinear connection N on E and used it to obtain the algebra of Finsler tensor fields on E. Next, he has defined the torsion and curvature of Finsler connection and he has given them with local components.

After that, the geometry of contact and paracontact structures on vector bundles with the aid of Finsler connections has been studied in [2,3] and [4]. In these studies, the contact and paracontact Finsler structures on E have been given by the Finsler tensor field Φ of type $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, a 1-form η and a vector field ξ. Later the metric structure g on E has been decomposed as $g = g_h + g_v$ and some characterizations of the structure (Φ, η, ξ, g) have been given. As a conclusion, in [4] it is proved that, the Riemannian Sasakian structures and Sasakian Finsler structures are adaptable.

In this paper, the tensors $N^{(1)}$, $N^{(2)}$, $N^{(3)}$ and $N^{(4)}$ which characterize the condition of normality of the almost paracontact Finsler structure are defined on an almost paracontact Finsler manifold (E, Φ, η, ξ) with the help of Nijenhuis tensor N_Φ and some characterizations for these tensors are given. Later, the paracontact metric Finsler manifolds and K-paracontact Finsler manifolds are defined and it is shown that the necessary and sufficient condition for a paracontact metric Finsler structure to be K-paracontact Finsler structure is $N^{(3)} = 0$. Since $N^{(3)}$ gives important results for a paracontact Finsler structure, we define a tensor field $\mathcal{H}$ which is a symmetric operator and anti-commutative with Φ. The para-Sasakian Finsler manifold is defined and given the condition for an almost paracontact metric Finsler structure to be para-Sasakian. It is proved that, the flag curvature of a plane which contains ξ is equal to -1 at each point of E which is a K-paracontact Finsler manifold. Some characterizations for paracontact structures with the aid of the curvature of the paracontact Finsler manifolds and the Ricci tensor of a para-Sasakian Finsler manifold are obtained. Finally, some results for the Ricci semi-symmetric para-Sasakian Finsler manifolds and para-Sasakian Finsler manifolds with η-parallel Ricci tensors are given.

2 Preliminaries

Let $\xi = (E, \pi, M)$ be a vector bundle of the class C^∞, where E is the total space of dimension $(n+m)$, M is the base space of dimension n and the local fibre $E_p = \pi^{-1}(p)$, $p \in M$, is a real vector space of dimension n.

The map $\pi : E \to M$ induces the π^T-morphism of the corresponding tangent bundles $\pi^T : T(E) \to T(M)$. Then $VE = Ker\pi^T$ is a subbundle of $T(E)$ called the *vertical bundle*. VE defines a distribution

$$E^V : u \in E \to E_u^V,$$

where E_u^V is the fibre of VE in the point $u \in E$ and E^V is called the *vertical distribution* of ξ. On the open set $\pi^{-1}(U_\alpha)$, $\frac{\partial}{\partial y^a}$, $a = 1, ..., m$, is a local basis of the vertical distribution E^V. Hence, E^V is integrable.

A non-linear connection on the total space E of ξ is a differentiable distribution $N : u \in E \to N_u \subset E_u$, with the property

$$E_u = N_u \oplus E_u^V,$$

where E_u is the tangent space at $u \in E$ to the manifold E. Here N_u is called the *horizontal distribution.* So,

Proposition 2.1. *If the base space M of the vector bundle ξ is paracompact, then there exist the non-linear connections on E [1].*

For every vector field X on E, there exists a unique decomposition as

$$X = X_h + X_v, \;\; X_{u_h} \in N_u, \; X_{u_v} \in E_u^V, \; \forall u \in E.$$

Here, X_h and X_v are called the *horizontal part* and the *vertical part* of X, respectively (see [1]).

Let x^i, $i = 1, 2, ..., n$ and y^a, $a = 1, 2, ..., m$ be the coordinates of $u \in E$. The local base of N_u is

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^a(x, y)\frac{\partial}{\partial y^a}$$

and that of E_u^V is $\frac{\partial}{\partial y^a}$, where N_i^a are coefficients of N. Their dual bases are $(dx^i, \delta y^a)$, where

$$\delta y^a = dy^a + N_i^a(x, y)dx^i.$$

Since these bases are dual, we have

$$\left\langle \frac{\delta}{\delta x^i}, dx^j \right\rangle = \delta_i^j, \ \left\langle \frac{\delta}{\delta x^i}, \delta y^a \right\rangle = 0, \ \left\langle \frac{\partial}{\partial y^a}, dx^j \right\rangle = 0, \ \left\langle \frac{\partial}{\partial y^a}, \delta y^b \right\rangle = \delta_a^b.$$

If $X = X^i(x, y)\frac{\delta}{\delta x^i} + \tilde{X}^a(x, y)\frac{\partial}{\partial y^a}, \forall X \in T_u(E)$, then

$$X_h = X^i(x, y)\frac{\delta}{\delta x^i}, \ X_v = \tilde{X}^a(x, y)\frac{\partial}{\partial y^a}, \ \tilde{X}^a = X^a + N_i^a X^i$$

and if $w = \tilde{w}_i dx^i + w_a \delta y^a$ is a 1-form, then

$$w_h = \tilde{w}_i dx^i, \ \tilde{w}_i = w_i - N_i^a w_a, \ w_v = w_a \delta y^a.$$

So it gives that, $w_h(X_v) = 0$ and $w_v(X_h) = 0$, where $w = w_h + w_v$ [3].

Definition 2.1. A tensor field t on the total space E of the vector bundle ξ is called a Finsler tensor field of the type $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$, if it has the property

$$\begin{aligned} & t(w_1, ..., w_p, X_1, ..., X_q, w_{p+1}, ..., w_{p+r}, X_{q+1}, ..., X_{q+s}) \\ & = t((w_1)_h, ..., (w_p)_h, (X_1)_h, ..., (X_q)_h, (w_{p+1})_v, ..., (w_{p+r})_v, (X_{q+1})_v, ..., (X_{q+s})_v), \end{aligned}$$

$\forall w_\alpha \in \chi^*(E), \forall X_\beta \in \chi(E)$ [1].

Proposition 2.2. *A Finsler tensor field of the type* $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$ *on* E *has the following local form [1]:*

$$t = t^{l_1,...,l_p,a_1,...,a_r}_{j_1,...,j_q,b_1,...,b_s}(x, y)\frac{\delta}{\delta x^{l_1}} \otimes ... \otimes \frac{\delta}{\delta x^{l_p}} \otimes dx^{j_1} \otimes ... \otimes dx^{j_q} \otimes \frac{\partial}{\partial y^{a_1}} \otimes ... \otimes \frac{\partial}{\partial y^{a_r}} \otimes \delta y^{b_1} \otimes ... \otimes \delta y^{b_s}.$$

Definition 2.2. A Finsler connection on E is a linear connection ∇ on E with the property that the horizontal linear spaces $N_u, u \in E$, of the distribution N are parallel with respect to ∇ and similarly, the vertical linear spaces $E_u^V, u \in E$, are parallel with respect to ∇ [1].

A Finsler connection ∇ on E is characterized by the conditions

$$(\nabla_X Y_h)_v = 0 \ \text{ and } \ (\nabla_X Y_v)_h = 0, \ \forall X, Y \in \chi(E).$$

Thus we have:

Theorem 2.1. *The following statements are equivalent:*
(a) ∇ *is a Finsler connection on* E,
(b) $\nabla_X Y = (\nabla_X Y_h)_h + (\nabla_X Y_v)_v, \forall X, Y \in \chi(E)$,
(c) $\nabla_X w = (\nabla_X w_h)_h + (\nabla_X w_v)_v, \forall w \in \chi^*(E), \forall X \in \chi(E)$ *[1].*

If $\underset{F}{\mathfrak{F}}{}^{pr}_{qs}(E)$ is the $\mathfrak{F}(E)$-module of the Finsler tensor fields of the type $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$, then the $\mathfrak{F}(E)$-module

$$\underset{F}{\mathfrak{F}}(E) = \underset{p,q,r,s=0,1,\dots F}{\oplus} \mathfrak{F}{}^{pr}_{qs}(E)$$

and the product tensor is a graded algebra called the algebra of Finsler tensor fields on E.

For a Finsler connection ∇ on E

$$\nabla^h_X Y = \nabla_{X_h} Y,\ \nabla^v_X Y = \nabla_{X_v} Y,\ \forall X, Y \in \chi(E),$$

where ∇^h and ∇^v are the covariant derivatives in the algebra $\underset{F}{\mathfrak{F}}(E)$. Here, ∇^h is called the h-*covariant derivative* and ∇^v is called the v-*covariant derivative* of the Finsler connection ∇. Furthermore, we have

(i) $\nabla^h_X f = X_h(f)$, $(\nabla^h_X Y_h)_v = 0$, $(\nabla^h_X Y_v)_h = 0$,
(ii) $\nabla^h_X Y = (\nabla^h_X Y_h)_h + (\nabla^h_X Y_v)_v$,
(iii) $\nabla^v_X f = X_v(f)$, $(\nabla^v_X Y_h)_v = 0$, $(\nabla^v_X Y_v)_h = 0$,
(iv) $\nabla^v_X Y = (\nabla^v_X Y_h)_h + (\nabla^v_X Y_v)_v$

and we have analogous formulas for $\nabla_X w$, too (see [1]).

The torsion tensor field T of a Finsler connection ∇ on E is given by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],\ \forall X, Y \in \chi(E),$$

which is characterized by the five Finsler tensor fields:

$$[T(X_h,Y_h)]_h,\ [T(X_h,Y_h)]_v,\ [T(X_h,Y_v)]_h,\ [T(X_h,Y_v)]_v,\ [T(X_v,Y_v)]_v.$$

If the Finsler connection ∇ on E is without torsion, then we have

$$T(X_h,Y_h) = 0,\ T(X_h,Y_v) = 0,\ T(X_v,Y_v) = 0,\ \forall X, Y \in \chi(E) \text{ [3]}.$$

If we break T down into horizontal and vertical parts, we have

$$\begin{aligned} &T_h(X_h,Y_h) = \nabla^h_X Y_h - \nabla^h_Y X_h - [X_h,Y_h]_h,\ T_v(X_h,Y_h) = -[X_h,Y_h]_v, \\ &T_h(X_h,Y_v) = -\nabla^v_Y X_h - [X_h,Y_v]_h,\ T_v(X_h,Y_v) = \nabla^h_X Y_v - [X_h,Y_v]_v, \\ &T_v(X_v,Y_v) = \nabla^v_X Y_v - \nabla^v_Y X_v - [X_v,Y_v]_v \end{aligned}$$

and when the Finsler connection ∇ is torsion free, we get

$$\begin{aligned} &[X_h,Y_h]_h = \nabla^h_X Y_h - \nabla^h_Y X_h,\ [X_h,Y_h]_v = 0,\ [X_h,Y_v]_h = -\nabla^v_Y X_h, \\ &[X_h,Y_v]_v = \nabla^h_X Y_v,\ [X_v,Y_v]_v = \nabla^v_X Y_v - \nabla^v_Y X_v \text{[5]}. \end{aligned}$$

If w is a differential r-form on E and ∇ is a linear connection on E, then the *exterior differential* dw is given by

$$\begin{aligned} dw(X_0, X_1, ..., X_r) = &\frac{1}{r+1}\sum_{i=0}^{r}(-1)^i X_i(w(X_0, ..., \hat{X}_i, ..., X_r)) \\ &+ \frac{1}{r+1}\sum_{0\le i<j\le r}(-1)^{i+j} w([X_i,X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_r), \end{aligned}$$

where $\hat{X}_k$ means that the term X_k is omitted. If w is a 1-form, then

$$2dw(X,Y) = X(w(Y)) - Y(w(X)) - w([X,Y]) \text{ [6]}.$$

Thus, if $w \in \chi^*(E)$ is a 1-form and ∇ is a Finsler connection on E, then for $\forall X, Y \in \chi(E)$ we have

$$\left.\begin{aligned} 2dw(X_h,Y_h) = X_h(w(Y_h)) - Y_h(w(X_h)) - w([X_h,Y_h]), \\ 2dw(X_v,Y_h) = X_v(w(Y_h)) - Y_h(w(X_v)) - w([X_v,Y_h]), \\ 2dw(X_v,Y_v) = X_v(w(Y_v)) - Y_v(w(X_v)) - w([X_v,Y_v]). \end{aligned}\right\} \tag{2.1}$$

3 Almost Paracontact Finsler Structures on Vector Bundle

A (4n+2)-dimensional Finsler manifold $E^{(4n+2)}$ has an *almost paracontact Finsler structure* (Φ, η, ξ), if it admits a tensor field Φ of type $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, a vector field ξ and a 1-form η satisfying the following conditions [2]:

$$\left.\begin{array}{c} \Phi.\Phi = I_n - \eta_h \otimes \xi_h - \eta_v \otimes \xi_v, \\ \Phi\xi_h = 0,\ \Phi\xi_v = 0, \\ \eta_h(\xi_h) + \eta_v(\xi_v) = 1, \\ \eta_h(\Phi X_h) = \eta_v(\Phi X_h) = \eta_h(\Phi X_v) = \eta_v(\Phi X_v) = 0. \end{array}\right\} \tag{3.1}$$

Let g be the pseudo-Riemannian Finsler metric on E. Then the metric structure g on E can be decomposed as

$$g = g_h + g_v, \tag{3.2}$$

where g_h is of type $\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$ and g_v is of type $\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$. Thus, we have

$$\begin{aligned} g(X,Y) &= g_h(X,Y) + g_v(X,Y) \\ &= g(X_h,Y_h) + g(X_v,Y_v),\ \forall X,Y \in \chi(E) \text{ [2]}. \end{aligned} \tag{3.3}$$

Furthermore, the Finsler connection ∇ with respect to g is given by

$$\begin{aligned} 2g_h(\nabla^h_X Y_h, Z_h) &= X_h g_h(Y_h,Z_h) + Y_h g_h(X_h,Z_h) - Z_h g_h(X_h,Y_h) \\ &\quad + g_h([X_h,Y_h],Z_h) + g_h([Z_h,X_h],Y_h) - g_h([Y_h,Z_h],X_h), \end{aligned} \tag{3.4}$$

$$\begin{aligned} 2g_v(\nabla^v_X Y_v, Z_v) &= X_v g_v(Y_v,Z_v) + Y_v g_v(X_v,Z_v) - Z_v g_v(X_v,Y_v) \\ &\quad + g_v([X_v,Y_v],Z_v) + g_v([Z_v,X_v],Y_v) - g_v([Y_v,Z_v],X_v), \end{aligned} \tag{3.5}$$

$$2g_h(\nabla^v_X Y_h, Z_h) = X_v g_h(Y_h,Z_h) + g_h([X_v,Y_h]_h, Z_h) + g_h([Z_h,X_v]_h, Y_h), \tag{3.6}$$

$$2g_v(\nabla^h_X Y_v, Z_v) = X_h g_v(Y_v,Z_v) + g_v([X_h,Y_v]_v, Z_v) + g_v([Z_v,X_h]_v, Y_v). \tag{3.7}$$

If the Finsler manifold $E^{(4n+2)}$ with (Φ, η, ξ)-structure admits a pseudo-Riemannian Finsler metric g such that

$$g_h(\Phi X, \Phi Y) = -g_h(X,Y) + \eta_h(X)\eta_h(Y) \text{ and } g_v(\Phi X, \Phi Y) = -g_v(X,Y) + \eta_v(X)\eta_v(Y), \tag{3.8}$$

which is equivalent to

$$\left.\begin{array}{l} (i)\ g_h(X,\xi) = \eta_h(X),\ g_v(X,\xi) = \eta_v(X), \\ (ii)\ g_h(\Phi X, \Phi Y) = -g_h(\Phi^2 X, Y),\ g_v(\Phi X, \Phi Y) = -g_v(\Phi^2 X, Y), \end{array}\right\} \tag{3.9}$$

then we say that $E^{(4n+2)}$ has an *almost paracontact metric Finsler structure* and g is called *compatible metric* (see [2]).

Now, we define

$$\Omega(X,Y) = g(X,\Phi Y);\ \Omega(X_h,Y_h) = g_h(X,\Phi Y),\ \Omega(X_v,Y_v) = g_v(X,\Phi Y) \tag{3.10}$$

and call it the *fundamental 2-form*. Then, for $\forall X,Y \in \chi(E)$, the fundamental 2-form satisfies

$$\begin{aligned} &(i)\ \Omega(\Phi X_h, \Phi Y_h) = -\Omega(X_h,Y_h),\ \Omega(\Phi X_v, \Phi Y_v) = -\Omega(X_v,Y_v), \\ &(ii)\ \Omega(X_h,Y_h) = -\Omega(Y_h,X_h),\ \Omega(X_v,Y_v) = -\Omega(Y_v,X_v). \end{aligned} \tag{3.11}$$

Also we have the following equations with respect to the fundamental 2-form Ω:

$$\begin{aligned} d\Omega(X_h,Y_h,Z_h) &= X_h\Omega(Y_h,Z_h)+Y_h\Omega(Z_h,X_h)+Z_h\Omega(X_h,Y_h)\\ &\quad -\Omega([X_h,Y_h],Z_h)-\Omega([Z_h,X_h],Y_h)-\Omega([Y_h,Z_h],X_h), \end{aligned} \tag{3.12}$$

$$\begin{aligned} d\Omega(X_v,Y_v,Z_v) &= X_v\Omega(Y_v,Z_v)+Y_v\Omega(Z_v,X_v)+Z_v\Omega(X_v,Y_v)\\ &\quad -\Omega([X_v,Y_v],Z_v)-\Omega([Z_v,X_v],Y_v)-\Omega([Y_v,Z_v],X_v), \end{aligned} \tag{3.13}$$

$$d\Omega(X_v,Y_h,Z_h) = X_v\Omega(Y_h,Z_h)-\Omega([X_v,Y_h]_h,Z_h)-\Omega([Z_h,X_v]_h,Y_h), \tag{3.14}$$

$$d\Omega(X_v,Y_v,Z_h) = Z_h\Omega(X_v,Y_v)-\Omega([Z_h,X_v]_v,Y_v)-\Omega([Y_v,Z_h]_v,X_v), \tag{3.15}$$

$$d\Omega(X_h,Y_v,Z_h) = Y_v\Omega(Z_h,X_h)-\Omega([X_h,Y_v]_h,Z_h)-\Omega([Y_v,Z_h]_h,X_h), \tag{3.16}$$

$$d\Omega(X_h,Y_v,Z_v) = X_h\Omega(Y_v,Z_v)-\Omega([X_h,Y_v]_v,Z_v)-\Omega([Z_v,X_h]_v,Y_v), \tag{3.17}$$

$$d\Omega(X_v,Y_h,Z_v) = Y_h\Omega(Z_v,X_v)-\Omega([X_v,Y_h]_v,Z_v)-\Omega([Y_h,Z_v]_v,X_v), \tag{3.18}$$

$$d\Omega(X_h,Y_h,Z_v) = Z_v\Omega(X_h,Y_h)-\Omega([Z_v,X_h]_h,Y_h)-\Omega([Y_h,Z_v]_h,X_h). \tag{3.19}$$

Example 3.1. *Let $\xi=(E,\pi,M)$ be a vector bundle of the class C^∞, where $E=\mathbb{R}^6$ is the total space of dimension 6 and $M=\mathbb{R}^3$ is the base space of dimension 3.*

If x_1,x_2,x_3 and y_1,y_2,y_3 are the coordinates of $u=(x,y)\in E$, then the local base of N_u is $(\frac{\delta}{\delta x_1},\frac{\delta}{\delta x_2},\frac{\delta}{\delta x_3})$ and the local base of E_u^V is $(\frac{\partial}{\partial y_1},\frac{\partial}{\partial y_2},\frac{\partial}{\partial y_3})$, such that $E_u=N_u\oplus E_u^V$.

Let the vector field $\mathcal{X}$ on E be

$$\mathcal{X}=\mathcal{X}_h+\mathcal{X}_v=\underbrace{X\frac{\delta}{\delta x_1}+Y\frac{\delta}{\delta x_2}+Z\frac{\delta}{\delta x_3}}_{\mathcal{X}_h}+\underbrace{\tilde{X}\frac{\partial}{\partial y_1}+\tilde{Y}\frac{\partial}{\partial y_2}+\tilde{Z}\frac{\partial}{\partial y_3}}_{\mathcal{X}_v},$$

the 1-form η be

$$\eta=\eta_h+\eta_v=\underbrace{x_2dx_1-2dx_3+\frac{x_2X}{Y}dx_2}_{\eta_h}+\underbrace{y_2\delta y_1-2\delta y_3+\frac{y_2\tilde{X}}{\tilde{Y}}\delta y_2}_{\eta_v},$$

where $\eta_h(\mathcal{X}_v)=0$ and $\eta_v(\mathcal{X}_h)=0$.

The structure vector field ξ is given by

$$\xi=\xi_h+\xi_v=\underbrace{\frac{1}{2x_2}\frac{\delta}{\delta x_1}}_{\xi_h}+\underbrace{\frac{1}{2y_2}\frac{\partial}{\partial y_1}}_{\xi_v}.$$

The tensor field Φ of type $\begin{pmatrix}1&1\\1&1\end{pmatrix}$ by the matris form is

$$\Phi=\begin{bmatrix}0&\frac{1}{x_2}&0&0&0&0\\0&0&1&0&0&0\\0&1&0&0&0&0\\0&0&0&0&\frac{1}{y_2}&0\\0&0&0&0&0&1\\0&0&0&0&1&0\end{bmatrix},\ \textit{where}\ \Phi_h=\begin{bmatrix}0&\frac{1}{x_2}&0\\0&0&1\\0&1&0\end{bmatrix}\ \textit{and}\ \Phi_v=\begin{bmatrix}0&\frac{1}{y_2}&0\\0&0&1\\0&1&0\end{bmatrix}$$

are tensor fields of type $(1,1)$.

Then one can see that

$$\begin{aligned}&\Phi^2\mathcal{X}=\mathcal{X}-\eta_h(\mathcal{X}_h)\xi_h-\eta_v(\mathcal{X}_v)\xi_v,\\ &\Phi_h(\xi_h)=0,\ \Phi_v(\xi_v)=0,\\ &\eta_h(\xi_h)+\eta_v(\xi_v)=1,\ \eta_h(\xi_v)=0,\ \eta_v(\xi_h)=0,\\ &\eta_h(\Phi_h(\mathcal{X}_h))=\eta_v(\Phi_v(\mathcal{X}_v))=0,\end{aligned}$$

where $Y^2 = x_2XZ$ *and* $\tilde{Y}^2 = y_2\tilde{X}\tilde{Z}$.

Therefore, (Φ, η, ξ) *is an almost paracontact Finsler structure on* $E = \mathbb{R}^6$.

4 Normal Almost Paracontact Finsler Manifolds

The *Nijenhuis tensor* N_j of a tensor field J of type (1,1) on a manifold M is a tensor field of type (1,2) defined by

$$N_j(X,Y) = J^2\left[X,Y\right] + \left[JX,JY\right] - J\left[JX,Y\right] - J\left[X,JY\right],\ \forall X,Y \in TM. \tag{4.1}$$

If M admits a tensor field J of type (1,1) satisfying

$$J^2 = I,$$

then it is said to be an *almost product manifold* equipped with an *almost product structure* J. An almost product structure is *integrable* if its Nijenhuis tensor vanishes [7].

Let $M^{(2n+1)}$ be an almost paracontact manifold with structure (Φ, η, ξ) and consider the manifold $M^{(2n+1)} \times \mathbb{R}$. We denote a vector field on $M^{(2n+1)} \times \mathbb{R}$ by $(X, f\frac{d}{dt})$, where X is tangent to $M^{(2n+1)}$, t is coordinate on $\mathbb{R}$ and f is a C^∞-function on $M^{(2n+1)} \times \mathbb{R}$. For any two vector fields $(X, f\frac{d}{dt})$ and $(Y, h\frac{d}{dt})$, one can see that

$$\left[(X, f\frac{d}{dt}), (Y, h\frac{d}{dt})\right] = (\left[X,Y\right], (Xh - Yf)\frac{d}{dt}). \tag{4.2}$$

An almost paracomplex structure J on $M^{(2n+1)} \times \mathbb{R}$ is defined by

$$J(X, f\frac{d}{dt}) = (\Phi X + f\xi, \eta(X)\frac{d}{dt}). \tag{4.3}$$

Here one can easily see that $J^2 = I$.

If J is integrable, we say that the almost paracontact structure (Φ, η, ξ) is *normal*.

As the vanishing of the Nijenhuis tensor of J is necessary and sufficient condition for integrability, we express the condition of normality in terms of Nijenhuis tensor N_Φ of Φ. Since N_j is tensor field of type (1,2), it suffices to compute $N_j((X,0),(Y,0))$ and $N_j((X,0),(0,\frac{d}{dt}))$. So from (4.1), (4.2) and (4.3), we get

$$\begin{aligned} N_j((X,0),(Y,0)) &= (N_\Phi(X,Y) - 2d\eta(X,Y)\xi, ((L_{\Phi X}\eta)Y - (L_{\Phi Y}\eta)X)\frac{d}{dt}), \\ N_j((X,0),(0,\frac{d}{dt})) &= -((L_\xi \Phi)X, (L_\xi \eta)X\frac{d}{dt}). \end{aligned}$$

We are thus led to define tensors $N^{(1)}, N^{(2)}, N^{(3)}$ and $N^{(4)}$ by

$$\left.\begin{array}{c} N^{(1)}(X,Y) = N_\Phi(X,Y) - 2d\eta(X,Y)\xi, \\ N^{(2)}(X,Y) = (L_{\Phi X}\eta)Y - (L_{\Phi Y}\eta)X, \\ N^{(3)}(X) = (L_\xi\Phi)X, \\ N^{(4)}(X) = (L_\xi\eta)X. \end{array}\right\} \tag{4.4}$$

So, the almost paracontact structure (Φ, η, ξ) is normal if and only if these four tensors vanish (see [8]).

Now let us define these tensors, found above, on almost paracontact Finsler manifold (E, Φ, η, ξ):

For $\forall X_h, Y_h, \xi_h \in N_u$ and $\forall X_v, Y_v, \xi_v \in E_u^V$,

$$\begin{aligned} N^{(1)}(X_h, Y_h) &= N_\Phi(X_h, Y_h) - 2d\eta_h(X_h, Y_h)\xi_h \\ &= [X_h, Y_h] + [\Phi X_h, \Phi Y_h] - \Phi\,[\Phi X_h, Y_h] \\ &\quad - \Phi\,[X_h, \Phi Y_h] - X_h(\eta_h(Y_h))\xi_h + Y_h(\eta_h(X_h))\xi_h, \\ N^{(2)}(X_h, Y_h) &= (L_{\Phi X_h}\eta_h)Y_h - (L_{\Phi Y_h}\eta_h)X_h \\ &= \Phi X_h \eta_h(Y_h) - \eta_h([\Phi X_h, Y_h]) - \Phi Y_h \eta_h(X_h) + \eta_h([\Phi Y_h, X_h]), \\ N^{(3)}(X_h) &= (L_{\xi_h}\Phi)(X_h) = [\xi_h, \Phi X_h] - \Phi\,[\xi_h, X_h]\,, \\ N^{(4)}(X_h) &= (L_{\xi_h}\eta_h)(X_h) = \xi_h \eta_h(X_h) - \eta_h([\xi_h, X_h]); \end{aligned} \tag{4.5}$$

$$\begin{aligned} N^{(1)}(X_v, Y_v) &= N_\Phi(X_v, Y_v) - 2d\eta_v(X_v, Y_v)\xi_v, \\ N^{(2)}(X_v, Y_v) &= (L_{\Phi X_v}\eta_v)Y_v - (L_{\Phi Y_v}\eta_v)X_v, \\ N^{(3)}(X_v) &= (L_{\xi_v}\Phi)(X_v), \\ N^{(4)}(X_v) &= (L_{\xi_v}\eta_v)(X_v); \end{aligned} \tag{4.6}$$

$$\begin{aligned} N^{(1)}(X_v, Y_h) &= N_\Phi(X_v, Y_h) - 2d\eta_h(X_v, Y_h)\xi_h - 2d\eta_v(X_v, Y_h)\xi_v \\ &= [X_v, Y_h] + [\Phi X_v, \Phi Y_h] - \Phi\,[\Phi X_v, Y_h] \\ &\quad - \Phi\,[X_v, \Phi Y_h] - X_v(\eta_h(Y_h))\xi_h + Y_h(\eta_v(X_v))\xi_v, \\ N^{(2)}(X_v, Y_h) &= (L_{\Phi X_v}\eta_h)Y_h + (L_{\Phi X_v}\eta_v)Y_h - (L_{\Phi Y_h}\eta_h)X_v - (L_{\Phi Y_h}\eta_v)X_v \\ &= \Phi X_v \eta_h(Y_h) - \eta_h([\Phi X_v, Y_h]) - \eta_v([\Phi X_v, Y_h]) \\ &\quad + \eta_h([\Phi Y_h, X_v]) - \Phi Y_h \eta_v(X_v) + \eta_v([\Phi Y_h, X_v]), \\ N^{(3)}(X_v) &= (L_{\xi_h}\Phi)(X_v),\ N^{(4)}(X_v) = (L_{\xi_h}\eta_v)(X_v), \\ N^{(3)}(Y_h) &= (L_{\xi_v}\Phi)(Y_h),\ N^{(4)}(Y_h) = (L_{\xi_v}\eta_h)(Y_h). \end{aligned} \tag{4.7}$$

So, the almost paracontact Finsler structure (Φ, η, ξ) is normal if and only if the tensors $N^{(1)}, N^{(2)}, N^{(3)}$ and $N^{(4)}$ vanish identically.

Proposition 4.1. *Let E be an almost paracontact Finsler manifold with an almost paracontact Finsler structure (Φ, η, ξ). Then we have,*

$$N^{(1)}(X_h, \xi_h) = -N^{(3)}(\Phi X_h) = -\,[\xi_h, X_h] + \Phi\,[\xi_h, \Phi X_h] + \xi_h(\eta_h(X_h))\xi_h, \tag{4.8}$$

$$N^{(2)}(X_h, Y_h) = 2(d\eta_h(\Phi X_h, Y_h) + d\eta_h(X_h, \Phi Y_h)), \tag{4.9}$$

$$N^{(4)}(X_h) = 2d\eta_h(\xi_h, X_h); \tag{4.10}$$

$$N^{(1)}(X_v, \xi_v) = -N^{(3)}(\Phi X_v) = -\,[\xi_v, X_v] + \Phi\,[\xi_v, \Phi X_v] + \xi_v(\eta_v(X_v))\xi_v, \tag{4.11}$$

$$N^{(2)}(X_v, Y_v) = 2(d\eta_v(\Phi X_v, Y_v) + d\eta_v(X_v, \Phi Y_v)), \tag{4.12}$$

$$N^{(4)}(X_v) = 2d\eta_v(\xi_v, X_v); \tag{4.13}$$

$$N^{(1)}(X_v,\xi_h) = -N^{(3)}(\Phi X_v) + \eta_v(X_v)\,[\xi_v,\xi_h]$$
$$= -\Phi\,[\Phi X_v,\xi_h] + [X_v,\xi_h] + \xi_h(\eta_v(X_v))\xi_v, \tag{4.14}$$
$$N^{(1)}(Y_h,\xi_v) = -N^{(3)}(\Phi Y_h) + \eta_h(Y_h)\,[\xi_h,\xi_v]$$
$$= -\Phi\,[\Phi Y_h,\xi_v] + [Y_h,\xi_v] + \xi_v(\eta_h(Y_h))\xi_h, \tag{4.15}$$
$$N^{(2)}(X_v,Y_h) = 2(d\eta_h(\Phi X_v,Y_h) + d\eta_v(\Phi X_v,Y_h)$$
$$+ d\eta_h(X_v,\Phi Y_h) + d\eta_v(X_v,\Phi Y_h)), \tag{4.16}$$
$$N^{(2)}(X_h,Y_v) = 2(d\eta_h(\Phi X_h,Y_v) + d\eta_v(\Phi X_h,Y_v)$$
$$+ d\eta_h(X_h,\Phi Y_v) + d\eta_v(X_h,\Phi Y_v)), \tag{4.17}$$
$$N^{(4)}(Y_h) = 2d\eta_h(\xi_v,Y_h),\ N^{(4)}(X_v) = 2d\eta_v(\xi_h,X_v). \tag{4.18}$$

Proof. For $\forall X_h, Y_h, \xi_h \in N_u$ and $\forall X_v, Y_v, \xi_v \in E_u^V$, from (2.1), (3.1), (4.5), (4.6) and (4.7), after necessary calculations we obtain (4.8)-(4.18). ☐

So, from (3.1), (4.9), (4.10), (4.12), (4.13), (4.16), (4.17) and (4.18) we have:

Corollary 4.1. *If E is an almost paracontact Finsler manifold with an almost paracontact Finsler structure (Φ,η,ξ), then for $\forall X_h, Y_h, \xi_h \in N_u$ and $\forall X_v, Y_v, \xi_v \in E_u^V$, we have*

$$N^{(2)}(X_h,\Phi Y_h) = 2(d\eta_h(\Phi X_h,\Phi Y_h) + d\eta_h(X_h,Y_h)) + \eta_h(Y_h)N^{(4)}(X_h), \tag{4.19}$$
$$N^{(2)}(X_v,\Phi Y_v) = 2(d\eta_v(\Phi X_v,\Phi Y_v) + d\eta_v(X_v,Y_v)) + \eta_v(Y_v)N^{(4)}(X_v), \tag{4.20}$$
$$N^{(2)}(X_v,\Phi Y_h) = 2(d\eta_h(\Phi X_v,\Phi Y_h) + d\eta_v(\Phi X_v,\Phi Y_h) + d\eta_h(X_v,Y_h) + d\eta_v(X_v,Y_h))$$
$$+ \eta_h(Y_h)N^{(4)}(X_v) + \eta_h(Y_h)2d\eta_h(\xi_h,X_v), \tag{4.21}$$
$$N^{(2)}(X_h,\Phi Y_v) = 2(d\eta_h(\Phi X_h,\Phi Y_v) + d\eta_v(\Phi X_h,\Phi Y_v) + d\eta_h(X_h,Y_v) + d\eta_v(X_h,Y_v))$$
$$+ \eta_v(Y_v)N^{(4)}(X_h) + \eta_v(Y_v)2d\eta_v(\xi_v,X_h). \tag{4.22}$$

Proposition 4.2. *On an almost paracontact Finsler manifold E, $N^{(2)}$ vanishes if and only if*

$$d\eta(\Phi X,\Phi Y) = -d\eta(X,Y). \tag{4.23}$$

Proof. Let us assume that $N^{(2)} = 0$. Then from (4.10) and (4.19), for $\forall X_h, Y_h, \xi_h \in N_u$,

$$0 = N^{(2)}(X_h,\Phi Y_h) = 2(d\eta_h(\Phi X_h,\Phi Y_h) + d\eta_h(X_h,Y_h)) + \eta_h(Y_h)2d\eta_h(\xi_h,X_h)$$

and for this, it must be $d\eta_h(\Phi X_h,\Phi Y_h) = -d\eta_h(X_h,Y_h)$.

Similarly, from (4.13) and (4.20), if $N^{(2)} = 0$ for $\forall X_v, Y_v, \xi_v \in E_u^V$, then it must be $d\eta_v(\Phi X_v,\Phi Y_v) = -d\eta_v(X_v,Y_v)$.

Furthermore, from (4.18) and (4.21), if

$$0 = N^{(2)}(X_v,\Phi Y_h) = 2(d\eta_h(\Phi X_v,\Phi Y_h) + d\eta_v(\Phi X_v,\Phi Y_h) + d\eta_h(X_v,Y_h) + d\eta_v(X_v,Y_h))$$
$$+ \eta_h(Y_h)2d\eta_h(\xi_h,X_v) + \eta_h(Y_h)2d\eta_v(\xi_h,X_v),$$

then $d\eta_h(\Phi X_v,\Phi Y_h) = -d\eta_h(X_v,Y_h)$ and $d\eta_v(\Phi X_v,\Phi Y_h) = -d\eta_v(X_v,Y_h)$. Finally, from (4.22), if $N^{(2)}(X_h,\Phi Y_v) = 0$, then it must be $d\eta_h(\Phi X_h,\Phi Y_v) = -d\eta_h(X_h,Y_v)$ and $d\eta_v(\Phi X_h,\Phi Y_v) = -d\eta_v(X_h,Y_v)$.

Conversely, let us take into account that $d\eta(\Phi X,\Phi Y) = -d\eta(X,Y)$. Then, for $\forall X_h, Y_h, \xi_h \in N_u$, we have

$$0 = d\eta_h(\Phi^2 X_h,\Phi\xi_h) = -d\eta_h(\Phi X_h,\xi_h).$$

So, from this equality and (3.1), we get

$$d\eta_h(\Phi X_h,\Phi^2 Y_h) = -d\eta_h(X_h,\Phi Y_h) \implies d\eta_h(\Phi X_h,Y_h) = -d\eta_h(X_h,\Phi Y_h).$$

Using the last equation in (4.9), we obtain $N^{(2)}(X_h, Y_h) = 0$.

Similarly, for $\forall X_v, Y_v, \xi_v \in E_u^V$, from $0 = d\eta_v(\Phi^2 X_v, \Phi\xi_v) = -d\eta_v(\Phi X_v, \xi_v)$ and (3.1), we have

$$d\eta_v(\Phi X_v, Y_v) = -d\eta_v(X_v, \Phi Y_v).$$

Using this expression in (4.12), we obtain that $N^{(2)}(X_v, Y_v) = 0$.

Now, if $d\eta_h(\Phi X_v, \Phi Y_h) = -d\eta_h(X_v, Y_h)$, then using $0 = d\eta_h(\Phi^2 X_v, \Phi\xi_h) = -d\eta_h(\Phi X_v, \xi_h)$, we get

$$d\eta_h(\Phi X_v, Y_h) = -d\eta_h(X_v, \Phi Y_h) \tag{4.24}$$

and if $d\eta_v(\Phi X_v, \Phi Y_h) = -d\eta_v(X_v, Y_h)$, then using $0 = d\eta_v(\Phi^2 X_v, \Phi\xi_h) = -d\eta_v(\Phi X_v, \xi_h)$, we get

$$d\eta_v(\Phi X_v, Y_h) = -d\eta_v(X_v, \Phi Y_h). \tag{4.25}$$

Thus, from (4.16), (4.24) and (4.25) we obtain that $N^{(2)}(X_v, Y_h) = 0$.

Lastly, if we take $d\eta_h(\Phi X_h, \Phi Y_v) = -d\eta_h(X_h, Y_v)$ and $d\eta_v(\Phi X_h, \Phi Y_v) = -d\eta_v(X_h, Y_v)$, we get $d\eta_h(\Phi X_h, Y_v) = -d\eta_h(X_h, \Phi Y_v)$ and $d\eta_v(\Phi X_h, Y_v) = -d\eta_v(X_h, \Phi Y_v)$, respectively. So, using the last two equations in (4.17), we can see that $N^{(2)}(X_h, Y_v) = 0$. □

Lemma 4.2. *If $N^{(1)} = 0$ for an almost paracontact Finsler manifold E, then $N^{(2)} = N^{(3)} = N^{(4)} = 0$.*

Proof. If $N^{(1)} = 0$, from (4.5), for $\forall X_h, Y_h, \xi_h \in N_u$, we get

$$-\Phi[\Phi X_h, \xi_h] + [X_h, \xi_h] + \xi_h(\eta_h(X_h))\xi_h = 0 \tag{4.26}$$

and applying η_h to (4.26) we obtain that

$$\xi_h(\eta_h(X_h)) - \eta_h([\xi_h, X_h]) = (L_{\xi_h}\eta_h)(X_h) = N^{(4)}(X_h) = 0. \tag{4.27}$$

Replacing X_h by ΦX_h in (4.27), we have

$$\eta_h([\xi_h, \Phi X_h]) = 0. \tag{4.28}$$

Applying Φ to (4.26) and using (4.28), we obtain that $N^{(3)}(X_h) = 0$.

From $N^{(1)} = 0$ and (4.5),

$$\begin{aligned} 0 = N^{(1)}(\Phi X_h, Y_h) &= [X_h, \Phi Y_h] - \eta_h(X_h)[\xi_h, \Phi Y_h] + \Phi Y_h(\eta_h(X_h))\xi_h \\ &- \Phi[X_h - \eta_h(X_h)\xi_h, Y_h] - \Phi[\Phi X_h, \Phi Y_h] + [\Phi X_h, Y_h] - \Phi X_h(\eta_h(Y_h))\xi_h \end{aligned}$$

and applying η_h to the last equation, we obtain that $N^{(2)}(X_h, Y_h) = 0$.

Analogously, for $\forall X_v, Y_v, \xi_v \in E_u^V$, if $N^{(1)}(X_v, Y_v) = 0$, then we can see that $N^{(2)}(X_v, Y_v) = 0$, $N^{(3)}(X_v) = 0$ and $N^{(4)}(X_v) = 0$.

If $N^{(1)}(X_v, Y_h) = 0$, from (4.7) we have

$$-\Phi[\Phi X_v, \xi_h] + [X_v, \xi_h] + \xi_h(\eta_v(X_v))\xi_v = 0 \tag{4.29}$$

and applying η_h and η_v to (4.29), we get

$$\eta_h([X_v, \xi_h]) = 0 \text{ and } \eta_v([X_v, \xi_h]) = -\xi_h(\eta_v(X_v)). \tag{4.30}$$

So, from (4.30) we have $N^{(4)}(X_v) = 0$.

If we apply Φ to (4.29), then we have

$$-[\Phi X_v, \xi_h] + \eta_h([\Phi X_v, \xi_h])\xi_h + \eta_v([\Phi X_v, \xi_h])\xi_v + \Phi[X_v, \xi_h] = 0. \tag{4.31}$$

Replacing X_v by ΦX_v in (4.30) and using this in (4.31), we obtain that $N^{(3)}(X_v) = 0$.

Also, from (4.7) we have

$$0 = N^{(1)}(\xi_v, Y_h) = -\Phi[\xi_v, \Phi Y_h] + [\xi_v, Y_h] - \xi_v(\eta_h(Y_h))\xi_h \tag{4.32}$$

and applying η_h and η_v to (4.32), we obtain that

$$\eta_h([\xi_v, Y_h]) = \xi_v(\eta_h(Y_h)) \text{ and } \eta_v([\xi_v, Y_h]) = 0. \tag{4.33}$$

So, from (4.33) it can be seen that $N^{(4)}(Y_h) = 0$.

Applying Φ to (4.32) and using (4.33), we get $N^{(3)}(Y_h) = 0$.

Finally, from $N^{(1)}(\Phi X_v, Y_v) = 0$, we get

$$\begin{aligned}&[X_v, \Phi Y_h] - \eta_v(X_v)[\xi_v, \Phi Y_h] + \Phi Y_h(\eta_v(X_v))\xi_v - \Phi[X_v - \eta_v(X_v)\xi_v, Y_h] \\ &- \Phi[\Phi X_v, \Phi Y_h] + [\Phi X_v, Y_h] - \Phi X_v(\eta_h(Y_h))\xi_h = 0\end{aligned} \tag{4.34}$$

and applying η_v and η_h to (4.34), we have

$$\eta_v([X_v, \Phi Y_h]) + \Phi Y_h \eta_v(X_v) + \eta_v([\Phi X_v, Y_h]) = 0 \tag{4.35}$$

and

$$\eta_h([X_v, \Phi Y_h]) + \eta_h([\Phi X_v, Y_h]) - \Phi X_v \eta_h(Y_h) = 0, \tag{4.36}$$

respectively. So, from (4.35) and (4.36) we obtain that $N^{(2)}(X_v, Y_h) = 0$. □

Thus we have:

Corollary 4.3. *An almost paracontact Finsler structure (Φ, η, ξ) is normal if and only if $N^{(1)} = 0$.*

5 Paracontact Metric Finsler Manifolds

Definition 5.1. If $\Omega = d\eta$, that is,

$$g_h(X, \Phi Y) = d\eta_h(X_h, Y_h) \text{ and } g_v(X, \Phi Y) = d\eta_v(X_v, Y_v), \tag{5.1}$$

then η is a paracontact form and the almost paracontact metric Finsler manifold (E, Φ, η, ξ, g) is said to be *paracontact metric Finsler manifold.*

Lemma 5.1. *For a paracontact metric Finsler structure (Φ, η, ξ, g), $N^{(2)}$ and $N^{(4)}$ vanish. Furthermore, $N^{(3)}$ vanishes if and only if ξ is a Killing vector field.*

Proof. If the structure (Φ, η, ξ, g) is paracontact metric Finsler, then from (4.5) and (5.1), for $\forall X_h, Y_h, \xi_h \in N_u$, we get

$$N^{(4)}(X_h) = (L_{\xi_h}\eta_h)(X_h) = \xi_h(\eta_h(X_h)) - \eta_h([\xi_h, X_h]) = 2d\eta_h(\xi_h, X_h) = 2g_h(\xi_h, \Phi X_h) = 0$$

and similarly from (4.6), for $\forall X_v, Y_v, \xi_v \in E_u^V$, we get $N^{(4)}(X_v) = 0$. Furthermore, from (4.7) we have

$$N^{(4)}(X_v) = (L_{\xi_h}\eta_v)(X_v) = 2d\eta_v(\xi_h, X_v) = 2g_v(\xi_h, \Phi X_v) = 0$$

and

$$N^{(4)}(Y_h) = (L_{\xi_v}\eta_h)(Y_h) = 2d\eta_h(\xi_v, Y_h) = 2g_h(\xi_v, \Phi Y_h) = 0.$$

On the other hand, from (3.9), (4.5) and (5.1), we get $N^{(2)}(X_h, Y_h) = 0$; from (3.9), (4.6) and (5.1), we get $N^{(2)}(X_v, Y_v) = 0$ and also from (4.7), we obtain that $N^{(2)}(X_v, Y_h) = 0$.

Since

$$(L_{\xi_h}g)(X_h, \xi_h) = (L_{\xi_h}\eta_h)(X_h) = 0$$

and η and $d\eta$ are invariant under Lie derivation, we get $L_{\xi_h}d\eta_h = 0$. So, for $\forall X_h, Y_h \in N_u$, we get

$$0 = (L_{\xi_h}d\eta_h)(X_h, Y_h) = \xi_h(d\eta_h(X_h, Y_h)) - d\eta_h([\xi_h, X_h], Y_h) - d\eta_h(X_h, [\xi_h, Y_h])$$

and from (5.1)

$$\xi_h(g(X_h, \Phi Y_h)) - g([\xi_h, X_h], \Phi Y_h) - g(X_h, \Phi[\xi_h, Y_h]) = 0. \quad (5.2)$$

On the other hand, if we sum up the equations

$$(L_{\xi_h} g)(X_h, \Phi Y_h) = \xi_h(g(X_h, \Phi Y_h)) - g([\xi_h, X_h], \Phi Y_h) - g(X_h, [\xi_h, \Phi Y_h])$$
$$g(X_h, (L_{\xi_h}\Phi)Y_h) = g(X_h, [\xi_h, \Phi Y_h]) - g(X_h, \Phi[\xi_h, Y_h])$$

and use (5.2), then we get

$$(L_{\xi_h} g)(X_h, \Phi Y_h) + g(X_h, (L_{\xi_h}\Phi)Y_h) = 0.$$

Thus, if $N^{(3)} = 0$, then $(L_{\xi_h} g)(X_h, \Phi Y_h) = 0$. Since this equation is true for $\forall X_h, Y_h \in N_u$, we have $L_{\xi_h} g = 0$ and therefore, ξ_h is a Killing vector field.

Conversely, if ξ_h is a Killing vector field, then we have $g(X_h, N^{(3)}(Y_h)) = 0$ and since it is satisfied for $\forall X_h, Y_h \in N_u$, we get $N^{(3)} = 0$.

Similar computations can be done for $\forall X_v, Y_v, \xi_v \in E_u^V$. So, this completes the proof. □

Definition 5.2. A paracontact metric Finsler structure (Φ, η, ξ, g) with ξ Killing vector field is called a *K-paracontact Finsler structure*.

So, we can give the following corollary:

Corollary 5.2. *A paracontact metric Finsler structure (Φ, η, ξ, g) is K-paracontact if and only if $N^{(3)}$ vanishes.*

Proposition 5.1. *For an almost paracontact metric Finsler structure (Φ, η, ξ, g) on E, the covariant derivative of Φ with respect to the Finsler connection ∇ is given by*

$$\begin{aligned}2g((\nabla_X\Phi)Y, Z) = &-d\Omega(X, Y, Z) - d\Omega(X, \Phi Y, \Phi Z) - g(N^{(1)}(Y, Z), \Phi X) \\ &+ N^{(2)}(Y, Z)\eta(X) - 2d\eta(\Phi Z, X)\eta(Y) + 2d\eta(\Phi Y, X)\eta(Z),\end{aligned} \quad (5.3)$$

where Ω is the fundamental 2 form.

Furthermore, if the structure (Φ, η, ξ, g) is paracontact metric Finsler, then the equation (5.3) simplifies to

$$2g((\nabla_X\Phi)Y, Z) = -g(N^{(1)}(Y, Z), \Phi X) - 2d\eta(\Phi Z, X)\eta(Y) + 2d\eta(\Phi Y, X)\eta(Z). \quad (5.4)$$

Proof. From (3.4) and (3.10), for $\forall X_h, Y_h, Z_h, \xi_h \in N_u$ we have

$$\begin{aligned}2g_h((\nabla^h_X\Phi)Y_h, Z_h) = &\Phi Y_h g_h(X_h, Z_h) - Z_h\Omega(X_h, Y_h) + g_h([X_h, \Phi Y_h], Z_h) + \Omega([Z_h, X_h], Y_h) \\ &- g_h([\Phi Y_h, Z_h], X_h) + Y_h\Omega(X_h, Z_h) - \Phi Z_h g_h(X_h, Y_h) \\ &+ \Omega([X_h, Y_h], Z_h) + g_h([\Phi Z_h, X_h], Y_h) - g_h([Y_h, \Phi Z_h], X_h).\end{aligned}$$

Furthermore, from (3.12) we have

$$\begin{aligned}&d\Omega(X_h, \Phi Y_h, \Phi Z_h) = -X_h\Omega(Y_h, Z_h) - \Phi Y_h g_h(Z_h, X_h) + \Phi Y_h(\eta_h(X_h)\eta_h(Z_h)) \\ &+ \Phi Z_h g_h(X_h, Y_h) - \Phi Z_h(\eta_h(X_h)\eta_h(Y_h)) - g_h([X_h, \Phi Y_h], Z_h) + \eta_h([X_h, \Phi Y_h])\eta_h(Z_h) \\ &- g_h([\Phi Z_h, X_h], Y_h) + \eta_h([\Phi Z_h, X_h])\eta_h(Y_h) - \Omega([\Phi Y_h, \Phi Z_h], X_h);\end{aligned} \quad (5.5)$$

from (3.10) and (4.5) we get

$$\begin{aligned}&g(N^{(1)}(Y_h, Z_h), \Phi X_h) = \Omega([\Phi Y_h, \Phi Z_h], X_h) + \Omega([Y_h, Z_h], X_h) + g_h([\Phi Y_h, Z_h], X_h) \\ &- \eta_h(X_h)\eta_h([\Phi Y_h, Z_h]) + g_h([Y_h, \Phi Z_h], X_h) - \eta_h(X_h)\eta_h([Y_h, \Phi Z_h])\end{aligned} \quad (5.6)$$

and

$$N^{(2)}(Y_h,Z_h)\eta_h(X_h) = \Phi Y_h(\eta_h(Z_h))\eta_h(X_h) - \eta_h([\Phi Y_h, Z_h])\eta_h(X_h) \\ - \Phi Z_h(\eta_h(Y_h))\eta_h(X_h) + \eta_h([\Phi Z_h, Y_h])\eta_h(X_h). \tag{5.7}$$

So, from (5.5), (5.6) and (5.7) we can reach to the desired equation.

Similarly, from (3.5), (3.10),(3.13) and (4.6), for $\forall X_v, Y_v, Z_v, \xi_v \in E_u^V$, we can see that the equation (5.3) is satisfied.

Since

$$\begin{aligned} 2g_h((\nabla_X^v\Phi)Y_h, Z_h) &= g_h([X_v, \Phi Y_h], Z_h) + \Omega([Z_h, X_v]_h, Y_h) \\ &\quad + \Omega([X_v, Y_h]_h, Z_h) + g_h([\Phi Z_h, X_v]_h, Y_h), \\ d\Omega(X_v, \Phi Y_h, \Phi Z_h) &= -X_v\Omega(Y_h, Z_h) - g_h([X_v, \Phi Y_h]_h, Z_h) + \eta_h([X_v, \Phi Y_h]_h)\eta_h(Z_h) \\ &\quad - g_h([\Phi Z_h, X_v]_h, Y_h) + \eta_h([\Phi Z_h, X_v]_h)\eta_h(Y_h), \\ g_h(N^{(1)}(Y_h, Z_h), \Phi X_v) &= 0,\ N^{(2)}(Y_h, Z_h)\eta_h(X_v) = 0 \end{aligned}$$

and by using (3.14), we get

$$2g_h((\nabla_X^v\Phi)Y_h, Z_h) = -d\Omega(X_v, Y_h, Z_h) - d\Omega(X_v, \Phi Y_h, \Phi Z_h) - g_h(N^{(1)}(Y_h, Z_h), \Phi X_v) \\ + N^{(2)}(Y_h, Z_h)\eta_h(X_v) - 2d\eta_h(\Phi Z_h, X_v)\eta_h(Y_h) + 2d\eta_h(\Phi Y_h, X_v)\eta_h(Z_h).$$

With a similar way, we can see that

$$2g_v((\nabla_X^h\Phi)Y_v, Z_v) = -d\Omega(X_h, Y_v, Z_v) - d\Omega(X_h, \Phi Y_v, \Phi Z_v) - g_h(N^{(1)}(Y_v, Z_v), \Phi X_h) \\ + N^{(2)}(Y_v, Z_v)\eta_v(X_h) - 2d\eta_v(\Phi Z_v, X_h)\eta_v(Y_v) + 2d\eta_v(\Phi Y_v, X_h)\eta_v(Z_v).$$

Finally, the equation (5.4) follows from the equation (5.3), Definition 5.1 and Lemma 5.1. □

On a paracontact metric Finsler manifold, we know that, for $\forall X_h, Y_h, Z_h, \xi_h \in N_u$,

$$2g_h((\nabla_X^h\Phi)Y_h, Z_h) = -g_h(N^{(1)}(Y_h, Z_h), \Phi X_h) - 2d\eta_h(\Phi Z_h, X_h)\eta_h(Y_h) + 2d\eta_h(\Phi Y_h, X_h)\eta_h(Z_h). \tag{5.8}$$

Therefore, replacing X_h by ξ_h in this equation, we have

$$\nabla_\xi^h\Phi = 0 \tag{5.9}$$

and similarly from (5.4), for $\forall X_v, Y_v, Z_v, \xi_v \in E_u^V$, we have

$$\nabla_\xi^v\Phi = 0. \tag{5.10}$$

Furthermore, since $g_h(\xi, \xi) = 1$ for $\forall \xi_h \in N_u$, we get

$$g_h(\nabla_X^h\xi_h, \xi_h) = 0,\ \forall X_h \in N_u.$$

Thus, $g_h(\nabla_X^h\xi_h, \xi_h) = 0$ and since the Finsler connection ∇ is a metric connection we have

$$0 = \nabla_\xi^h(g_h(\xi_h, X_h)) = g_h(\nabla_\xi^h\xi_h, X_h) + g_h(\xi_h, \nabla_\xi^h X_h)$$

and so

$$\begin{aligned} g_h(\nabla_\xi^h\xi_h, X_h) = -g_h(\xi_h, \nabla_\xi^h X_h) &= -g_h(\xi_h, \nabla_X^h\xi_h + [\xi_h, X_h]) \\ &= -\eta_h([\xi_h, X_h]) = 2d\eta_h(\xi, X), \end{aligned}$$

where the vector field X_h is orthogonal to ξ_h. Since $d\eta_h(\xi, X) = 0$ on a paracontact metric Finsler manifold, we obtain that $g_h(\nabla_\xi^h\xi_h, X_h) = 0$ and so

$$\nabla_\xi^h\xi_h = 0. \tag{5.11}$$

Similarly for $\forall \xi_v \in E_u^V$, we can see that

$$\nabla_\xi^v \xi_v = 0. \tag{5.12}$$

From the Corollary 5.2, we know that a paracontact metric Finsler structure is K-paracontact if and only if $N^{(3)}$ vanishes. Since the tensor field $N^{(3)}$ gives important results for a paracontact Finsler structure, let we define a tensor field $\mathcal{H}$ on a paracontact Finsler manifold by

$$\mathcal{H} = \frac{1}{2} L_\xi \Phi = \frac{1}{2} N^{(3)}.$$

Thus, we have

$$\mathcal{H}\xi = \frac{1}{2} N^{(3)}(\xi) = \frac{1}{2}(L_\xi \Phi)(\xi) = \frac{1}{2}(L_{\xi_h + \xi_v}\Phi)(\xi_h + \xi_v) = 0. \tag{5.13}$$

Lemma 5.3. *The tensor field $\mathcal{H}$ on a paracontact metric Finsler manifold is a symmetric operator. Furthermore,*

$$\nabla_X^h \xi_h = -\Phi X_h + \Phi \mathcal{H} X_h, \tag{5.14}$$

$$\nabla_X^v \xi_v = -\Phi X_v + \Phi \mathcal{H} X_v, \tag{5.15}$$

$\mathcal{H}$ is anti-commutative with Φ and $tr\mathcal{H} = \mathcal{H}\xi = 0$.

Proof. On a paracontact metric Finsler manifold from (5.9) and (5.11), for $\forall X_h, Y_h, \xi_h \in N_u$, we have

$$g((L_{\xi_h}\Phi)X_h, Y_h) = g(-\nabla_{\Phi X}^h \xi_h + \Phi \nabla_X^h \xi_h, Y_h)$$

and here if we replace X_h or Y_h by ξ_h, then the equation vanishes.

Since $N^{(2)} = 0$ for a paracontact metric Finsler manifold, if we take X_h and Y_h are orthogonal to ξ_h, from (4.5), we obtain that

$$\eta_h([\Phi X_h, Y_h]) + \eta_h([X_h, \Phi Y_h]) = 0.$$

Also, since $-\Phi X_h(\eta_h(Y_h)) = X_h(\eta_h(\Phi Y_h)) = 0$ we have

$$-g_h(\nabla_{\Phi X}^h \xi_h, Y_h) - g_h(\nabla_X^h \xi_h, \Phi Y_h) = g_h(\nabla_X^h \Phi Y_h, \xi_h) + g_h(\nabla_{\Phi X}^h Y_h, \xi_h).$$

Hence we can see that $g(\mathcal{H}X_h, Y_h) = g(X_h, \mathcal{H}Y_h)$ and so $\mathcal{H}$ is symmetric.

Replacing Y_h by ξ_h in (5.8), we get

$$2g_h(-\Phi \nabla_X^h \xi_h, Z_h) = -g_h((L_{\xi_h}\Phi)X_h - 2X_h + 2\eta_h(X_h)\xi_h, Z_h)$$

and so

$$-\Phi \nabla_X^h \xi_h = -\frac{1}{2}(L_{\xi_h}\Phi)X_h + X_h - \eta_h(X_h)\xi_h.$$

Applying Φ to this equation, we obtain (5.14).

Now, let us see that $\mathcal{H}$ is anti-commutative with Φ:

$$2g_h(X_h, \Phi Y_h) = 2d\eta_h(X_h, Y_h) = g_h(X_h, \Phi Y_h) + g_h(\Phi \mathcal{H} X_h, Y_h) + g_h(X_h, \Phi Y_h) + g_h(\Phi X_h, \mathcal{H} Y_h)$$

and so

$$g_h(\Phi \mathcal{H} X_h, Y_h) + g_h(\Phi X_h, \mathcal{H} Y_h) = 0.$$

Since $\mathcal{H}$ is symmetric, we obtain that

$$\Phi \mathcal{H} + \mathcal{H} \Phi = 0. \tag{5.16}$$

If $\mathcal{H}X_h = \lambda X_h$, then $\mathcal{H}\Phi X_h = -\Phi \mathcal{H} X_h$ and so $\mathcal{H}\Phi X_h = -\lambda \Phi X_h$. Thus, if the eigenvalues of $\mathcal{H}$ are λ and $-\lambda$, then $trh = 0$.

With a similar way, the equation (5.15) and the same results can be obtained for $\forall X_v, Y_v, \xi_v \in E_u^V$, too. □

From the Corollary 5.2, we can give the following corollary:

Corollary 5.4. *Let (Φ, η, ξ, g) be a paracontact metric Finsler structure on E. Then (Φ, η, ξ, g) is a K-paracontact Finsler structure if and only if*

$$\nabla^h_X \xi_h = -\Phi X_h,\ \nabla^v_X \xi_v = -\Phi X_v. \tag{5.17}$$

Definition 5.3. If a paracontact metric Finsler manifold is normal, then it is called a *para-Sasakian Finsler manifold.*

Theorem 5.5. *An almost paracontact metric Finsler structure (Φ, η, ξ, g) is para-Sasakian if and only if*

$$\left.\begin{array}{c}(\nabla^h_X \Phi)Y_h = -g_h(X,Y)\xi_h + \eta_h(Y_h)X_h,\ (\nabla^v_X \Phi)Y_v = -g_v(X,Y)\xi_v + \eta_v(Y_v)X_v, \\ (\nabla^h_X \Phi)Y_v = 0,\ (\nabla^v_X \Phi)Y_h = 0.\end{array}\right\} \tag{5.18}$$

Moreover, a para-Sasakian Finsler manifold is K-paracontact.

Proof. Let us assume that the almost paracontact metric Finsler structure (Φ, η, ξ, g) is para-Sasakian. If the structure (Φ, η, ξ, g) is a normal paracontact metric Finsler structure then, we have $\Omega = d\eta$, $N^{(1)} = 0$ and $N^{(2)} = 0$. So from (3.8) and (5.4), for $\forall X_h, Y_h, Z_h, \xi_h \in N_u$ we get

$$(\nabla^h_X \Phi)Y_h = -g_h(X,Y)\xi_h + \eta_h(Y_h)X_h$$

and similarly for $\forall X_v, Y_v, Z_v, \xi_v \in E^V_u$, we obtain that

$$(\nabla^v_X \Phi)Y_v = -g_v(X,Y)\xi_v + \eta_v(Y_v)X_v.$$

Furthermore, again from (3.8) and (5.4) it is clear that $(\nabla^h_X \Phi)Y_v = 0$ and $(\nabla^v_X \Phi)Y_h = 0$.

Conversely, let us suppose that the almost paracontact metric Finsler structure (Φ, η, ξ, g) satisfies (5.18). For $\forall X_h, Y_h, \xi_h \in N_u$ replacing Y_h by ξ_h in the equation $(\nabla^h_X \Phi)Y_h = -g_h(X,Y)\xi_h + \eta_h(Y_h)X_h$, it is found that

$$-\Phi\nabla^h_X \xi_h = -\eta_h(X_h)\xi_h + X_h.$$

Applying Φ to this equation, we see that $\nabla^h_X \xi_h = -\Phi X_h$. Similarly, for $\forall X_v, Y_v, \xi_v \in E^V_u$, we get $\nabla^v_X \xi_v = -\Phi X_v$. So, from (5.17) ξ is Killing vector field. Thus, from

$$\begin{aligned}0 = (L_{\xi_h} g_h)(X_h, Y_h) &= \xi_h(g(X_h, Y_h)) - g([\xi_h, X_h], Y_h) - g(X_h, [\xi_h, Y_h]) \\ &= g(\nabla^h_X \xi_h, Y_h) + g(X_h, \nabla^h_Y \xi_h)\end{aligned}$$

we get

$$d\eta_h(X_h, Y_h) = g(\nabla^h_X \xi_h, Y_h) = \Omega(X_h, Y_h)$$

and similarly, $d\eta_v(X_v, Y_v) = \Omega(X_v, Y_v)$. Therefore, (Φ, η, ξ, g) is a paracontact metric Finsler structure. Furthermore, since

$$N_\Phi(X_h, Y_h) - 2d\eta_h(X_h, Y_h)\xi_h = 2g_h(\Phi Y_h, X_h)\xi_h - 2d\eta_h(X_h, Y_h)\xi_h = 0$$

and

$$N_\Phi(X_v, Y_v) - 2d\eta_v(X_v, Y_v)\xi_v = 2g_v(\Phi Y_v, X_v)\xi_v - 2d\eta_v(X_v, Y_v)\xi_v = 0$$

the paracontact metric Finsler structure (Φ, η, ξ, g) is a para-Sasakian Finsler structure. □

6 The Curvature of the Paracontact Finsler Manifold

The curvature of a Finsler connection ∇ is defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

such that

$$R(X,Y)Z = R_h(X,Y)Z_h + R_v(X,Y)Z_v,\ \forall X,Y,Z \in \chi(E)$$

and the operator $R(X,Y)$ is skew symmetric according to X and Y.

The curvature of a Finsler connection ∇ on $\chi(E)$ is defined by the following Finsler tensor fields [5]:

$$\left.\begin{array}{r}
R(X_h,Y_h)Z_h = \nabla^h_X \nabla^h_Y Z_h - \nabla^h_Y \nabla^h_X Z_h - \nabla_{[X_h,Y_h]} Z_h,\\
R(X_h,Y_h)Z_v = \nabla^h_X \nabla^h_Y Z_v - \nabla^h_Y \nabla^h_X Z_v - \nabla_{[X_h,Y_h]} Z_v,\\
R(X_v,Y_h)Z_h = \nabla^v_X \nabla^h_Y Z_h - \nabla^h_Y \nabla^v_X Z_h - \nabla_{[X_v,Y_h]} Z_h,\\
R(X_v,Y_h)Z_v = \nabla^v_X \nabla^h_Y Z_v - \nabla^h_Y \nabla^v_X Z_v - \nabla_{[X_v,Y_h]} Z_v,\\
R(X_v,Y_v)Z_h = \nabla^v_X \nabla^v_Y Z_h - \nabla^v_Y \nabla^v_X Z_h - \nabla_{[X_v,Y_v]} Z_h,\\
R(X_v,Y_v)Z_v = \nabla^v_X \nabla^v_Y Z_v - \nabla^v_Y \nabla^v_X Z_v - \nabla_{[X_v,Y_v]} Z_v.
\end{array}\right\} \tag{6.1}$$

Theorem 6.1. *Let E be a K-paracontact Finsler manifold of dimension (4n+2). Then, the flag curvature of a plane which contains ξ is equal to -1 at each point of E.*

Proof. If E is a K-paracontact Finsler manifold, then from (5.9), (5.17) and (6.1) we get

$$g_h(R(X_h,\xi_h)\xi_h, X_h) = -g_h(X_h,X_h) = -1,$$

where X_h is a unit vector field orthogonal to ξ_h.

Similarly, if X_v is a unit vector field orthogonal to ξ_v, we get

$$g_v(R(X_v,\xi_v)\xi_v, X_v) = -g_v(X_v,X_v) = -1.$$

Thus, we obtain the flag curvature of E as

$$K(X,\xi) = \frac{g_h(R(X_h,\xi_h)\xi_h, X_h) + g_v(R(X_v,\xi_v)\xi_v, X_v)}{g_h(X_h,X_h)g_h(\xi_h,\xi_h) + g_v(X_v,X_v)g_v(\xi_v,\xi_v)} = -1.$$

□

Proposition 6.1. *On a paracontact metric Finsler manifold E, we have*

$$\left.\begin{array}{c}
(\nabla^h_\xi \mathcal{H})X_h + (\nabla^v_\xi \mathcal{H})X_v = -\Phi X + \mathcal{H}^2\Phi X + \Phi R(\xi,X)\xi,\\
R(\xi,X)\xi + \Phi R(\xi,\Phi X)\xi = 2\Phi^2 X - 2\mathcal{H}^2 X.
\end{array}\right\} \tag{6.2}$$

Proof. Let E be a paracontact metric Finsler manifold. Then, from (5.9), (5.11), (5.14), (5.16) and (6.1), for $\forall X_h, \xi_h \in N_u$, we have

$$R(\xi_h,X_h)\xi_h = \Phi\nabla^h_\xi(\mathcal{H}X_h) - \Phi\nabla^h_\xi X_h - \Phi\mathcal{H}\,[\xi_h,X_h] + \Phi\,[\xi_h,X_h] \tag{6.3}$$

and applying Φ to this equation, we get

$$\Phi R(\xi_h,X_h)\xi_h = (\nabla^h_\xi\mathcal{H})X_h + \mathcal{H}\nabla^h_X\xi_h - \nabla^h_X\xi_h \tag{6.4}$$

$$= (\nabla^h_\xi\mathcal{H})X_h - \mathcal{H}^2\Phi X_h + \Phi X_h. \tag{6.5}$$

Replacing X by ΦX in (6.5) and using (5.9), (5.13), (5.16), we obtain

$$\Phi R(\xi_h,\Phi X_h)\xi_h = -\Phi(\nabla^h_\xi\mathcal{H})X_h - \mathcal{H}^2 X_h + \Phi^2 X_h. \tag{6.6}$$

Furthermore, from (6.3) we have

$$R(\xi_h, X_h)\xi_h = \Phi(\nabla^h_\xi \mathcal{H})X_h - \mathcal{H}^2 X_h + \Phi^2 X_h. \tag{6.7}$$

Thus, summing up (6.6) and (6.7) we get

$$R(\xi_h, X_h)\xi_h + \Phi R(\xi_h, \Phi X_h)\xi_h = 2\Phi^2 X_h - 2\mathcal{H}^2 X_h.$$

Similarly, for $\forall X_v, \xi_v \in E^V_u$, we can obtain that

$\Phi R(\xi_v, X_v)\xi_v = (\nabla^v_\xi \mathcal{H})X_v - \mathcal{H}^2\Phi X_v + \Phi X_v$ and $R(\xi_v, X_v)\xi_v + \Phi R(\xi_v, \Phi X_v)\xi_v = 2\Phi^2 X_v - 2\mathcal{H}^2 X_v.$

Hence, from the last equations we get

$$\Phi R(\xi, X)\xi = \Phi R_h(\xi, X)\xi_h + \Phi R_v(\xi, X)\xi_v = (\nabla^h_\xi \mathcal{H})X_h + (\nabla^v_\xi \mathcal{H})X_v + \Phi X - \mathcal{H}^2 \Phi X,$$

which is the first equation of (6.2) and

$$\begin{aligned} R(\xi, X)\xi + \Phi R(\xi, \Phi X)\xi &= R_h(\xi, X)\xi_h + R_v(\xi, X)\xi_v + \Phi R_h(\xi, \Phi X)\xi_h + \Phi R_v(\xi, \Phi X)\xi_v \\ &= 2\Phi^2 X - 2\mathcal{H}^2 X, \end{aligned}$$

which is the second equation of (6.2). □

Proposition 6.2. *On a para-Sasakian Finsler manifold, we have*

$$R(X, Y)\xi = \eta_h(X_h)Y_h + \eta_v(X_v)Y_v - \eta_h(Y_h)X_h - \eta_v(Y_v)X_v. \tag{6.8}$$

Proof. On a para-Sasakian Finsler manifold from (5.17) and (5.18), for $\forall X_h, Y_h, \xi_h \in N_u$, we get

$$\begin{aligned} R(X_h, Y_h)\xi_h &= \nabla^h_X \nabla^h_Y \xi_h - \nabla^h_Y \nabla^h_X \xi_h - \nabla_{[X_h, Y_h]}\xi_h \\ &= -(\nabla^h_X \Phi)Y_h + (\nabla^h_Y \Phi)X_h \\ &= \eta_h(X_h)Y_h - \eta_h(Y_h)X_h \end{aligned} \tag{6.9}$$

and similarly, for $\forall X_v, Y_v, \xi_v \in E^V_u$, we get

$$R(X_v, Y_v)\xi_v = \eta_v(X_v)Y_v - \eta_v(Y_v)X_v. \tag{6.10}$$

Thus, from (6.9) and (6.10), the equation (6.8) is obtained. □

Theorem 6.2. *Let ξ be a Killing vector field on a Finsler manifold E of dimension (4n+2). Then, E is a para-Sasakian Finsler manifold if and only if*

$$R(X, \xi)Y = g_h(X, Y)\xi_h + g_v(X, Y)\xi_v - \eta_h(Y_h)X_h - \eta_v(Y_v)X_v. \tag{6.11}$$

Proof. For $\forall X_h, Y_h, \xi_h \in N_u$, we get

$$\begin{aligned} &g(R_h(X_h, \xi_h)Y_h, X_h) = g(R_h(X_h, Y_h)\xi_h, X_h) \\ &= g(\nabla^h_X \nabla^h_Y \xi_h - \nabla^h_{\nabla^h_X Y_h}\xi_h, X_h) - g(\nabla^h_Y \nabla^h_X \xi_h - \nabla^h_{\nabla^h_Y X_h}\xi_h, X_h). \end{aligned} \tag{6.12}$$

Since ξ is a Killing vector field

$$0 = (L_{\xi_h} g_h)(X_h, Y_h) = g(\nabla^h_X \xi_h, Y_h) + g(X_h, \nabla^h_Y \xi_h)$$

and from here we have $g(\nabla^h_X \xi_h, X_h) = 0$. Differentiating this equation with respect to Y_h, we have

$$g_h(\nabla^h_Y \nabla^h_X \xi_h, X_h) + g_h(\nabla^h_X \xi_h, \nabla^h_Y X_h) = 0. \tag{6.13}$$

Again since ξ is a Killing vector field, we get

$$g(\nabla^h_{\nabla^h_Y X_h}\xi_h, X_h) + g_h(\nabla^h_Y X_h, \nabla^h_X \xi_h) = 0$$

and so the equation (6.13) is

$$g_h(\nabla^h_Y \nabla^h_X \xi_h - \nabla^h_{\nabla^h_Y X_h}\xi_h, X_h) = 0.$$

By using the last equation in (6.12), from (5.17) and (5.18) we obtain that

$$\begin{aligned} R_h(X,\xi)Y_h = R(X_h,\xi_h)Y_h &= -(\nabla^h_X \Phi)Y_h \\ &= g_h(X_h,Y_h)\xi_h - \eta_h(Y_h)X_h. \end{aligned} \tag{6.14}$$

Similarly, for $\forall X_v, Y_v, \xi_v \in E^V_u$, we get

$$R_v(X,\xi)Y_v = R(X_v,\xi_v)Y_v = g_v(X_v,Y_v)\xi_v - \eta_v(Y_v)X_v. \tag{6.15}$$

Taking into account (6.14) and (6.15), we have the desired equation. □

Therefore, we can give the following corollary:

Corollary 6.3. *On a para-Sasakian Finsler manifold, we have*

$$R(X,\xi)X = \xi,$$

where X_h and X_v are unit vector fields orthogonal to ξ_h and ξ_v, respectively.

Let $\{E^1_h, E^2_h, ..., E^{2n}_h, \xi_h\}$ and $\{E^1_v, E^2_v, ..., E^{2n}_v, \xi_v\}$ be the local orthogonal frames of N_u and E^V_u, respectively. Then, the *Ricci tensor* of a para-Sasakian Finsler manifold E of dimension (4n+2) is defined by

$$\begin{aligned} S(X,Y) = S_h(X,Y) + S_v(X,Y) &= S(X_h,Y_h) + S(X_v,Y_v) \\ &= \sum_{i=1}^{2n} g(R(X_h,E^i_h)E^i_h, Y_h) + g(R(X_h,\xi_h)\xi_h, Y_h) \\ &+ \sum_{i=1}^{2n} g(R(X_v,E^i_v)E^i_v, Y_v) + g(R(X_v,\xi_v)\xi_v, Y_v), \end{aligned} \tag{6.16}$$

where S_h and S_v denotes the *horizontal Ricci tensor* and the *vertical Ricci tensor* of E, respectively.

Lemma 6.4. *On a para-Sasakian Finsler manifold E of dimension (4n+2), we have*

$$S(X,\xi) = -2n\eta(X). \tag{6.17}$$

Proof. From (6.9), (6.10) and (6.16), for $\forall X_h, \xi_h \in N_u$, $\forall X_v, \xi_v \in E^V_u$ we have

$$\begin{aligned} S(X,\xi) = S_h(X,\xi) + S_v(X,\xi) &= \sum_{i=1}^{2n} g(R(X_h,E^i_h)E^i_h, \xi_h) + g(R(X_h,\xi_h)\xi_h, \xi_h) \\ &+ \sum_{i=1}^{2n} g(R(X_v,E^i_v)E^i_v, \xi_v) + g(R(X_v,\xi_v)\xi_v, \xi_v) \\ &= -2n\eta(X_h) - 2n\eta(X_v) = -2n\eta(X). \end{aligned}$$

□

Corollary 6.5. *On a (4n+2)-dimensional Finsler manifold E, a paracontact metric Finsler structure (Φ, η, ξ, g) is K-paracontact if and only if $S(\xi, \xi) = -2n$.*

From (6.17), we have

$$Q\xi = Q\xi_h + Q\xi_v = -2n\xi_h - 2n\xi_v = -2n\xi, \tag{6.18}$$

where Q is the *Ricci operator,* such that $S(X, Y) = g(QX, Y)$.

From (3.1), (3.9), (6.17) and the fact that $Q\Phi = \Phi Q$, for a para-Sasakian Finsler manifold E, we have

$$\begin{aligned} S(\Phi X, \Phi Y) &= S_h(\Phi X, \Phi Y) + S_v(\Phi X, \Phi Y) \\ &= g(Q\Phi X_h, \Phi Y_h) + g(Q\Phi X_v, \Phi Y_v) \\ &= -g(QX_h, Y_h) + \eta_h(Y_h)g(QX_h, \xi_h) - g(QX_v, Y_v) + \eta_v(Y_v)g(QX_v, \xi_v) \\ &= -S(X, Y) - 2n\{\eta_h(X_h)\eta_h(Y_h) + \eta_v(X_v)\eta_v(Y_v)\}. \end{aligned} \tag{6.19}$$

Proposition 6.3. *The curvature tensor of a paracontact metric Finsler manifold E satisfies*

$$\begin{aligned} R(\xi, X, Y, Z) &= -(\nabla^h_X \Omega)(Y_h, Z_h) + g(X_h, (\nabla^h_Y \Phi\mathcal{H})Z_h) - g(X_h, (\nabla^h_Z \Phi\mathcal{H})Y_h) \\ &\quad - (\nabla^v_X \Omega)(Y_v, Z_v) + g(X_v, (\nabla^v_Y \Phi\mathcal{H})Z_v) - g(X_v, (\nabla^v_Z \Phi\mathcal{H})Y_v). \end{aligned} \tag{6.20}$$

Proof. From (5.14), for $\forall X_h, Y_h, Z_h, \xi_h \in N_u$, we have

$$\begin{aligned} R_h(Y, Z)\xi_h = R(Y_h, Z_h)\xi_h &= \nabla^h_Y \nabla^h_Z \xi_h - \nabla^h_Z \nabla^h_Y \xi_h - \nabla_{[Y_h, Z_h]}\xi_h \\ &= (\nabla^h_Y \Phi\mathcal{H})Z_h - (\nabla^h_Z \Phi\mathcal{H})Y_h - (\nabla^h_Y \Phi)Z_h + (\nabla^h_Z \Phi)Y_h \end{aligned}$$

and from (3.10) and (5.18), we have

$$\begin{aligned} (\nabla^h_X \Omega)(Y_h, Z_h) &= \nabla^h_X(\Omega(Y_h, Z_h)) - \Omega(\nabla^h_X Y_h, Z_h) - \Omega(Y_h, \nabla^h_X Z_h) \\ &= g(Y_h, (\nabla^h_X \Phi)Z_h) = -\eta_h(Y_h)g(X_h, Z_h) + \eta_h(Z_h)g(X_h, Y_h). \end{aligned}$$

Thus, from (5.18), we obtain that

$$\begin{aligned} R(\xi_h, X_h, Y_h, Z_h) &= g(R(\xi_h, X_h)Y_h, Z_h) = g(R(Y_h, Z_h)\xi_h, X_h) \\ &= g((\nabla^h_Y \Phi\mathcal{H})Z_h, X_h) - g((\nabla^h_Z \Phi\mathcal{H})Y_h, X_h) - (\nabla^h_X \Omega)(Y_h, Z_h). \end{aligned}$$

Similarly, for $\forall X_v, Y_v, Z_v, \xi_v \in E^V_u$, we have

$$R(\xi_v, X_v, Y_v, Z_v) = g((\nabla^v_Y \Phi\mathcal{H})Z_v, X_v) - g((\nabla^v_Z \Phi\mathcal{H})Y_v, X_v) - (\nabla^v_X \Omega)(Y_v, Z_v).$$

Hence, from

$$\begin{aligned} R(\xi, X, Y, Z) &= g(R(\xi, X)Y, Z) = g(R(Y, Z)\xi, X) \\ &= g(R_h(Y, Z)\xi_h, X_h) + g(R_v(Y, Z)\xi_v, X_v) \end{aligned}$$

we have the equation (6.20). □

Proposition 6.4. *On a para-Sasakian Finsler manifold E*

$$\begin{aligned} &R(X, Y, \Phi Z, W) + R(X, Y, Z, \Phi W) \\ =&g(Y_h, Z_h)g(\Phi X_h, W_h) - g(Y_h, W_h)g(\Phi X_h, Z_h) - g(X_h, Z_h)g(\Phi Y_h, W_h) \\ &+ g(X_h, W_h)g(\Phi Y_h, Z_h) + g(Y_v, Z_v)g(\Phi X_v, W_v) - g(Y_v, W_v)g(\Phi X_v, Z_v) \\ &- g(X_v, Z_v)g(\Phi Y_v, W_v) + g(X_v, W_v)g(\Phi Y_v, Z_v) \end{aligned} \tag{6.21}$$

and

$$
\begin{aligned}
&R(\Phi X, \Phi Y, \Phi Z, \Phi W) - R(X, Y, Z, W)\\
=&\eta_h(X_h)\eta_h(W_h)g(Y_h, Z_h) + \eta_h(Y_h)\eta_h(Z_h)g(X_h, W_h) - \eta_h(Y_h)\eta_h(W_h)g(X_h, Z_h)\\
&- \eta_h(X_h)\eta_h(Z_h)g(Y_h, W_h) + \eta_v(X_v)\eta_v(W_v)g(Y_v, Z_v) + \eta_v(Y_v)\eta_v(Z_v)g(X_v, W_v)\\
&- \eta_v(Y_v)\eta_v(W_v)g(X_v, Z_v) - \eta_v(X_v)\eta_v(Z_v)g(Y_v, W_v). \qquad (6.22)
\end{aligned}
$$

Proof. For $\forall X_h, Y_h, Z_h, W_h, \xi_h \in N_u$

$$R(X_h, Y_h, \Phi Z_h, W_h) + R(X_h, Y_h, Z_h, \Phi W_h) = g(R(X_h, Y_h)\Phi Z_h, W_h) - g(R(X_h, Y_h)\Phi W_h, Z_h).$$

From (5.17) and (5.18), we have

$$
\begin{aligned}
R(X_h, Y_h)\Phi Z_h = &\, g(Y_h, Z_h)\Phi X_h - g(Z_h, \Phi X_h)Y_h - g(X_h, Z_h)\Phi Y_h\\
&+ g(Z_h, \Phi Y_h)X_h + \Phi R(X_h, Y_h)Z_h \qquad (6.23)
\end{aligned}
$$

and

$$
\begin{aligned}
R(X_h, Y_h)\Phi W_h = &\, g(Y_h, W_h)\Phi X_h - g(W_h, \Phi X_h)Y_h - g(X_h, W_h)\Phi Y_h\\
&+ g(W_h, \Phi Y_h)X_h + \Phi R(X_h, Y_h)W_h.
\end{aligned}
$$

Thus we get

$$
\begin{aligned}
R(X_h, Y_h, \Phi Z_h, W_h) + R(X_h, Y_h, Z_h, \Phi W_h) = &\, g(Y_h, Z_h)g(\Phi X_h, W_h)\\
&-g(Y_h, W_h)g(\Phi X_h, Z_h) - g(X_h, Z_h)g(\Phi Y_h, W_h) + g(X_h, W_h)g(\Phi Y_h, Z_h)
\end{aligned}
$$

and similarly, for $\forall X_v, Y_v, Z_v, W_v, \xi_v \in E_u^V$, we get

$$
\begin{aligned}
&R(X_v, Y_v, \Phi Z_v, W_v) + R(X_v, Y_v, Z_v, \Phi W_v) = g(Y_v, Z_v)g(\Phi X_v, W_v)\\
&- g(Y_v, W_v)g(\Phi X_v, Z_v) - g(X_v, Z_v)g(\Phi Y_v, W_v) + g(X_v, W_v)g(\Phi Y_v, Z_v).
\end{aligned}
$$

From the last two equations and

$$
\begin{aligned}
R(X, Y, \Phi Z, W) + R(X, Y, Z, \Phi W) &= g(R(X, Y)\Phi Z, W) + g(R(X, Y)Z, \Phi W)\\
&= g(R(X_h, Y_h)\Phi Z_h, W_h) + g(R(X_v, Y_v)\Phi Z_v, W_v)\\
&+ g(R(X_h, Y_h)Z_h, \Phi W_h) + g(R(X_v, Y_v)Z_v, \Phi W_v),
\end{aligned}
$$

(6.21) yields.

From (6.21), for $\forall X_h, Y_h, Z_h, W_h \in N_u$,

$$
\begin{aligned}
&R(\Phi X_h, \Phi Y_h, \Phi Z_h, \Phi W_h) + R(\Phi X_h, \Phi Y_h, Z_h, \Phi^2 W_h)\\
&= -g(\Phi X_h, \Phi^2 W_h)g(\Phi Y_h, Z_h) + g(\Phi X_h, \Phi Z_h)g(\Phi Y_h, \Phi W_h)\\
&- g(\Phi Y_h, \Phi Z_h)g(\Phi X_h, \Phi W_h) + g(\Phi Y_h, \Phi^2 W_h)g(\Phi X_h, Z_h)
\end{aligned}
$$

and from the last equation, (3.1), (3.8),(6.9) and (6.23), we have

$$
\begin{aligned}
&R(\Phi X_h, \Phi Y_h, \Phi Z_h, \Phi W_h) - R(X_h, Y_h, Z_h, W_h)\\
&= \eta_h(Y_h)\eta_h(Z_h)g(X_h, W_h) - \eta_h(Y_h)\eta_h(W_h)g(X_h, Z_h)\\
&- \eta_h(X_h)\eta_h(Z_h)g(Y_h, W_h) + \eta_h(X_h)\eta_h(W_h)g(Y_h, Z_h). \qquad (6.24)
\end{aligned}
$$

Similarly, for $\forall X_v, Y_v, Z_v, W_v \in E_u^V$ we get

$$
\begin{aligned}
&R(\Phi X_v, \Phi Y_v, \Phi Z_v, \Phi W_v) - R(X_v, Y_v, Z_v, W_v)\\
&= \eta_v(Y_v)\eta_v(Z_v)g(X_v, W_v) - \eta_v(Y_v)\eta_v(W_v)g(X_v, Z_v)\\
&- \eta_v(X_v)\eta_v(Z_v)g(Y_v, W_v) + \eta_v(X_v)\eta_v(W_v)g(Y_v, Z_v). \qquad (6.25)
\end{aligned}
$$

Taking into account (6.24) and (6.25), we obtain the equation (6.22). □

7 Ricci Semi-Symmetric Para-Sasakian Finsler Manifolds

Definition 7.1. A (4n+2)-dimensional Finsler manifold is called *horizontal Ricci semi-symmetric* (resp. *vertical Ricci semi-symmetric*), if the horizontal Ricci tensor S_h (resp. vertical Ricci tensor S_v) satisfies

$$R_h(X,Y)S_h = 0,\ \forall X_h, Y_h \in N_u \text{ (resp. } R_v(X,Y)S_v = 0,\ \forall X_v, Y_v \in E_u^V). \tag{7.1}$$

A Finsler manifold is said to be *Ricci semi-symmetric* if it is both horizontal and vertical Ricci semi-symmetric.

Definition 7.2. A Finsler manifold is said to be *Einstein Finsler manifold* if its Ricci tensor S is of the form

$$S(X,Y) = S_h(X,Y) + S_v(X,Y) = \lambda(g_h(X,Y) + g_v(X,Y)) = \lambda g(X,Y), \tag{7.2}$$

where λ is a constant.

Theorem 7.1. *A Ricci semi-symmetric para-Sasakian Finsler manifold is an Einstein Finsler manifold.*

Proof. Let us suppose that $R_h(X,Y)S_h = 0$ for $\forall X_h, Y_h \in N_u$. Then, from

$$(R_h(X,Y)S_h)(U_h,V_h) = -S_h(R_h(X,Y)U_h,V_h) - S_h(U_h,R_h(X,Y)V_h),\ \forall U_h,V_h \in N_u$$

we have

$$S_h(R_h(X,Y)U_h,V_h) + S_h(U_h,R_h(X,Y)V_h) = 0.$$

Replacing U_h by $\xi_h \in N_u$ in this equation, from (6.9) and (6.17), we get

$$\eta_h(X_h)S_h(Y_h,V_h) - \eta_h(Y_h)S_h(X_h,V_h) - 2n(\eta_h(Y_h)g(X_h,V_h) - \eta_h(X_h)g(Y_h,V_h)) = 0. \tag{7.3}$$

Putting $X_h = \xi_h$ in (7.3) and using (6.17), we obtain that

$$S_h(Y_h,V_h) = -2ng(Y_h,V_h).$$

Similarly, if we suppose that $R_v(X,Y)S_v = 0$ for $\forall X_v, Y_v \in E_u^V$, then we have

$$S_v(R_v(X,Y)U_v,V_v) + S_v(U_v,R_v(X,Y)V_v) = 0$$

and after the necessary calculations we obtain that

$$S_v(Y_v,V_v) = -2ng(Y_v,V_v).$$

So, we have

$$S(Y,V) = S_h(Y,V) + S_v(Y,V) = -2n(g_h(Y,V) + g_v(Y,V)) = -2ng(Y.V)$$

and therefore, a Ricci semi-symmetric para-Sasakian Finsler manifold is an Einstein Finsler manifold. □

8 Para-Sasakian Finsler Manifold with η-Parallel Ricci Tensor

Definition 8.1. The Ricci tensor S of a Finsler manifold E is called *η-parallel*, if it satisfies

$$(\nabla_X S)(\Phi Y, \Phi Z) = 0, \tag{8.1}$$

for all $X, Y, Z \in \chi(E)$.

Proposition 8.1. *A para-Sasakian Finsler manifold E has η-parallel Ricci tensor if and only if*

$$\begin{aligned}&(\nabla_X S)(Y,Z)\\&= 2n\{g(Y_h,\Phi X_h)\eta_h(Z_h)+g(Z_h,\Phi X_h)\eta_h(Y_h)+g(Y_v,\Phi X_v)\eta_v(Z_v)+g(Z_v,\Phi X_v)\eta_v(Y_v)\}\\&- \eta_h(Y_h)S_h(X_h,\Phi Z_h)-\eta_h(Z_h)S_h(\Phi Y_h,X_h)-\eta_v(Y_v)S_v(X_v,\Phi Z_v)-\eta_v(Z_v)S_v(\Phi Y_v,X_v).\end{aligned}$$

Proof. Let us suppose that the para-Sasakian Finsler manifold E has η-parallel Ricci tensor. Then, from (5.18), (6.17), (6.19) and (8.1), after long computations, we have

$$\begin{aligned}0 &= (\nabla_X S)(\Phi Y,\Phi Z) = (\nabla^h_X S_h)(\Phi Y_h,\Phi Z_h)+(\nabla^v_X S_h)(\Phi Y_h,\Phi Z_h)\\&+(\nabla^h_X S_v)(\Phi Y_v,\Phi Z_v)+(\nabla^v_X S_v)(\Phi Y_v,\Phi Z_v)\\&= \nabla^h_X(S_h(\Phi Y_h,\Phi Z_h))-S_h(\nabla^h_X(\Phi Y_h),\Phi Z_h)-S_h(\Phi Y_h,\nabla^h_X(\Phi Z_h))\\&+\nabla^v_X(S_h(\Phi Y_h,\Phi Z_h))-S_h(\nabla^v_X(\Phi Y_h),\Phi Z_h)-S_h(\Phi Y_h,\nabla^v_X(\Phi Z_h))\\&+\nabla^h_X(S_v(\Phi Y_v,\Phi Z_v))-S_v(\nabla^h_X(\Phi Y_v),\Phi Z_v)-S_v(\Phi Y_v,\nabla^h_X(\Phi Z_v))\\&+\nabla^v_X(S_v(\Phi Y_v,\Phi Z_v))-S_v(\nabla^v_X(\Phi Y_v),\Phi Z_v)-S_v(\Phi Y_v,\nabla^v_X(\Phi Z_v))\\&= -(\nabla^h_X S_h)(Y_h,Z_h)+2ng_h(Y_h,\Phi X_h)\eta_h(Z_h)+2n\eta_h(Y_h)g_h(Z_h,\Phi X_h)\\&-\eta_h(Y_h)S_h(X_h,\Phi Z_h)-\eta_h(Z_h)S_h(\Phi Y_h,X_h)\\&-(\nabla^v_X S_h)(Y_h,Z_h)-(\nabla^h_X S_v)(Y_v,Z_v)\\&-(\nabla^v_X S_v)(Y_v,Z_v)+2ng_v(Y_v,\Phi X_v)\eta_v(Z_v)+2n\eta_v(Y_v)g_v(Z_v,\Phi X_v)\\&-\eta_v(Y_v)S_v(X_v,\Phi Z_v)-\eta_v(Z_v)S_v(\Phi Y_v,X_v)\end{aligned}$$

and thus, we obtain that

$$\begin{aligned}&(\nabla^h_X S_h)(Y_h,Z_h)+(\nabla^v_X S_h)(Y_h,Z_h)+(\nabla^h_X S_v)(Y_v,Z_v)+(\nabla^v_X S_v)(Y_v,Z_v)\\&= 2n\{g(Y_h,\Phi X_h)\eta_h(Z_h)+g(Z_h,\Phi X_h)\eta_h(Y_h)+g(Y_v,\Phi X_v)\eta_v(Z_v)+g(Z_v,\Phi X_v)\eta_v(Y_v)\}\\&-\eta_h(Y_h)S_h(X_h,\Phi Z_h)-\eta_h(Z_h)S_h(\Phi Y_h,X_h)-\eta_v(Y_v)S_v(X_v,\Phi Z_v)-\eta_v(Z_v)S_v(\Phi Y_v,X_v),\end{aligned}\tag{8.2}$$

which completes the proof. □

Now, let $\{E^i_h, E^i_v\}$ be an orthonormal basis of $\chi(E)$. Putting $Y = Z = E^i$ in (8.2) and taking summation over the index i, we get

$$\sum_{i=1}^{2n+1}\{(\nabla^h_X S_h)(E^i_h,E^i_h)+(\nabla^v_X S_h)(E^i_h,E^i_h)+(\nabla^h_X S_v)(E^i_v,E^i_v)+(\nabla^v_X S_v)(E^i_v,E^i_v)\}=0.\tag{8.3}$$

Furthermore, the *scalar curvature* of a Finsler manifold is given by

$$r=\sum_{i=1}^{4n+2}S(E^i,E^i)=\sum_{i=1}^{2n+1}\{S_h(E^i,E^i)+S_v(E^i,E^i)\}.$$

Thus, from (8.3) we get

$$\begin{aligned}dr(X)=\nabla_X r &= \sum_{i=1}^{2n+1}\{\nabla_X(S_h(E^i,E^i))+\nabla_X(S_v(E^i,E^i))\}\\&=\sum_{i=1}^{2n+1}\{\nabla^h_X(S_h(E^i,E^i))+\nabla^v_X(S_h(E^i,E^i))+\nabla^h_X(S_v(E^i,E^i))+\nabla^v_X(S_v(E^i,E^i))\}\\&=\sum_{i=1}^{2n+1}\{(\nabla^h_X S_h)(E^i_h,E^i_h)+(\nabla^v_X S_h)(E^i_h,E^i_h)+(\nabla^h_X S_v)(E^i_v,E^i_v)+(\nabla^v_X S_v)(E^i_v,E^i_v)\}\\&=0.\end{aligned}$$

Hence, we have

Theorem 8.1. *The scalar curvature of para-Sasakian Finsler manifold E with η-parallel Ricci tensor is constant.*

9 Conclusion

Here, we give some characterizations about almost paracontact, almost paracontact metric, paracontact metric, K-paracontact and para-Sasakian Finsler structures on vector bundles and we give two classifications for para-Sasakian Finsler manifolds, which are useful for contact geometry.

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Competing Interests

The authors declare that no competing interests exist.

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