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Research Article

Combined Damped Sinusoidal Oscillation Solutions to the (3+1)-D Variable-Coefficient Generalized NLW Equation in Liquid with Gas Bubbles

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This paper investigates the combined damped sinusoidal oscillation solutions to the $(3+1)$-D variable-coefficient (VC) generalized nonlinear wave equation. The bilinear form is considered in terms of Hirota derivatives. Accordingly, we utilize a binary Bell polynomial transformation for reducing the Cole-Hopf algorithm to get the exact solutions of the VC generalized NLW equation. The damped sinusoidal oscillations for two cases of the nonlinear wave ordinary differential equation will be studied. Using suitable mathematical assumptions, the novel kinds of solitary, periodic, and singular soliton solutions are derived and established in view of the trigonometric and rational functions of the governing equation. To achieve this, the illustrative example of the VC generalized nonlinear wave equation is provided to demonstrate the feasibility and reliability of the procedure used in this study. The trajectory solutions of the traveling waves are shown explicitly and graphically. The effect of the free parameters on the behavior of acquired figures of a few obtained solutions for two nonlinear rational exact cases was also discussed. By comparing the proposed method with the other existing methods, the results show that the execution of this method is concise, simple, and straightforward.

1. Introduction

The advent of the concept of nonlinear partial differential equations (NLPDEs) has attracted the interest of many researchers due to their importance in accurately demonstrating the dynamics of abundant real-world systems in various fields of sciences such as physics, diffusion, biology, chaos theory, chemistry, engineering, economics, and commerce [1–3].

In particular, more and much attention has been paid to constructing exact and approximate solutions, for example, a multiobjective optimization structure method [4], the differential transform method [5], an improved differential transform method [6], a new solitary periodic wave solution [7], the multiple Exp-function method [8], an expectation maximization algorithm [9], a multiobjective mixed integer linear programming model [10], Hirota's bilinear method [11], a mean–semivariance approach [12], a genetic algorithm for preemptive scheduling of a single machine [13], solving absolute value problems [14], the inverse scattering transformation method [15], the multiple soliton solutions and fusion interaction phenomena [16], the truncated Painlevé series [17], a software product line engineering approach [18], a conceptual framework for SIDS alert system [19], the modified Pfaffian technique [20], the conserved quantities method [21], the Darboux transformation [22], and a computational intelligence approach [23].

In [24], the N-soliton solutions, soliton molecules, and asymmetric solitons of the Korteweg-de Vries-Caudrey-Dodd-Gibbon equation were obtained by means of the velocity resonance method by Ma et al. Researchers analyzed the higher-order algebraic soliton solutions of the Gerdjikov-Ivanov equation by using the Darboux transformation and some limit technique, and according to the asymptotic balance between different algebraic terms, they obtained the asymptotic expressions of algebraic soliton solutions [25]. Wang studied the multisoliton solutions of the $(2+1)$-dimensional PT-symmetric couplers with varying coefficients by using the homogeneous balance method [26]. The N-soliton solutions, M-breather solutions, and hybrid ones composed of solitons and breathers were constructed in a time-dependent KP equation by Wu [27]. Wang and Chen [28] investigated the higher-order Sawada-Kotera-type equation and the higher-order Lax-type equation in fluids. A $(2+1)$-dimensional coupled nonlinear partial differential equation with variable coefficients in an inhomogeneous medium according to the Hirota bilinear form and symbolic computation, the breather wave solutions and lump solutions were constructed by using the extended homoclinic breather technique and the generalized positive quadratic function method in Ref. [29]. Moreover, Sadat et al. [30] got lump-type solutions and their interaction solutions with one- or two-stripe solutions through the Hirota bilinear scheme and the Cole-Hopf transformation for a generalized $(3+1)$ shallow water-like equation. In addition, a kind of lump solution and two classes of interaction solutions were discussed to the $(2+1)$-dimensional generalized KdV equation with the aid of the symbolic computation system Mathematica and Hirota bilinear scheme [31].

The liquid with gas bubble problem has been made in the propagation of weakly nonlinear waves by Kudryashov and coauthors [32, 33]. In [34], a generalized $(3+1)$-dimensional ($(3+1)$-D) nonlinear wave (NLW) equation has been investigated to find the first-order lump wave solution and second-order lump wave solution which is given

$$(\Sigma_t + \phi_1 \Sigma\Sigma_x + \phi_2 \Sigma_{xxx} + \phi_3 \Sigma_x)_x + \phi_4 \Sigma_{yy} + \phi_5 \Sigma_{zz} = 0, \quad (1)$$

and also, $\phi_i \, (i = 1, \cdots, 5)$ is the constant coefficient. With $\phi_3 = 0$, the above equation is transformed to the $(3+1)$-dimensional generalized KP equation. N-soliton solutions and periodic wave solutions were studied for Equation (1) in liquid with gas bubbles in [35]. The $(3+1)$-D variable-coefficient (VC) generalized NLW equation [36] is taken as follows:

$$\begin{aligned}(\Sigma_t + \phi_1(t)\Sigma\Sigma x + \phi_2(t)\Sigma_{xxx} + \phi_3(t)\Sigma_x)_x \\ + \phi_4(t)\Sigma_{yy} + \phi_5(t)\Sigma_{zz} = 0,\end{aligned} \quad (2)$$

and also, $\phi_i(t) \, (i = 1, \cdots, 5)$ is the variable coefficient and $\Sigma = \Sigma(x, y, z, t)$ is the wave amplitude (unknown function) that should be searched. Equation (2) is a variable coefficient case of (1) that has been investigated in Refs. [34, 35]. The bilinear form, Bäcklund transformation, Lax pair, and infinitely many conservation laws were obtained via the binary Bell polynomials for a generalized $(3+1)$-D VC NLW equation by Deng and Gao [36]. In [37], for Equation (2), a periodic–shape lump solution, a parabolic–shape lump solution, a cubic–shape lump solution, and interaction solutions between lump and one solitary wave and between lump and two solitary waves were investigated. Also, Guo and Chen [38] studied the multisoliton solutions and periodic solutions including X–periodic, Y–periodic, and 2–periodic wave solutions. A few of rational exact solution for the $(3+1)$-D VC NLW equation has been studied in [39].

Hirota's bilinear method has always been a powerful tool for solving the N-soliton solution of the nonlinear evolution equation [40–42]. Since this method was proposed by Hirota, many scholars have continuously improved it to obtain the exact solution other than the N-soliton solution. Based on the bilinear method, Satsuma and Ablowitz proposed a long-wave limit method to obtain the lump solution, which decays algebraically in space [43]; Ohta and Yang found a way to get general rogue waves for Davey-Stewartson I system [44]; by means of bilinear approach, Lou discovered soliton solutions with even numbers and soliton solutions with odd numbers in nonlocal systems [45]; in recent years, the velocity resonance theory [46] and the resonance conditions mentioned in Refs. [47, 48] have further improved this method to better explain physical phenomena. Many researchers used various methods to study the nonlinear models by using the Hirota bilinear technique (HBT), such as a $(3+1)$-dimensional nonlinear evolution equation [49], the generalized variable-coefficient Kadomtsev–Petviashvili equation [50], the new $(3+1)$-dimensional generalized Kadomtsev–Petviashvili equation [51], and a generalized $(3+1)$-D VC NLW equation [52]. Through these methods, some exact solutions of the nonlinear models of equations were obtained. In order to really understand these physical phenomena, it is of immense importance to solve nonlinear partial differential equations (NLPDEs) which govern these aforementioned phenomena. However, there is no general systematic theory that can be applied to NLPDEs so that their analytic solutions can be obtained. Nevertheless, in recent times, scientists have developed effective techniques to obtain viable analytical solutions to NLPDEs, such as some nonlinear equations, for example, the $(3+1)$-dimensional BKP-Boussinesq equation [53], the generalized BKP equation [54], and new integrable Boussinesq equations of distinct dimensions [55]. In [56], the M-lump solution and N-soliton solution of the $(2+1)$-dimensional variable-coefficient Caudrey-Dodd-Gibbon-Kotera-Sawada equation were studied. In particular, more and much attention has been paid to constructing exact lump solutions to the third-order evolution equation [57] arising propagation of long waves over shallow water. In [58], a weight number was utilized in an algorithm to convey the Hirota condition while transforming the Hirota function in wave vectors to a homogeneous polynomial. Also, a generalized algorithm to prove the Hirota conditions was suggested by comparing degrees of the multivariate polynomials derived from the Hirota function in wave vectors by Ma [59]. In addition, N-soliton solutions in the case of nonlocal integrable equations have been presented by Riemann-Hilbert problems containing nonlocal reverse–space nonlinear Schrödinger hierarchies [60] and nonlocal real reverse–space time matrix AKNS hierarchies [61]. The simplified Hirota's technique with new complex forms was developed suitably to construct multiple–soliton solutions with a complex structure for modified KdV-Sine-Gordon equation in integrable form by Wazwaz and Kaur [62]. According to the bilinear forms, the novel N-soliton solutions of $(1+1)$ and $(2+1)$-dimensional generalized Broer–Kaup systems were obtained by using Hirota's bilinear method in [63].

Motivated by the above studies, we apply the proposed analytical method which is presented by [39] to solve the mentioned problems above. The advantage of the proposed method is that it can be applied to the integrable model and will be obtained plenty of combined damped sinusoidal oscillation solutions with three kinds of plotted graphs.

The rest of the paper is organized as follows: Section 2 presents the definition of the binary Bell polynomials and their properties. Also, the bilinear form of the $(3+1)$-D VC generalized NLW equation is constructed in Section 3. The proposed method HBT, for obtaining the combined damped sinusoidal oscillation solutions of Equation (2), is presented in Section 4. Finally, in Section 5, we briefly summarize and discuss the results in a conclusion.

2. Binary Bell Polynomials

Through Refs. [39, 64], take $\lambda=\lambda(x_1,x_2,\cdots,x_n)$ be a C^∞ function with multivariables, the general form can be written as

$$Y_{n_1x_1,\cdots,n_jx_j}(\lambda)\equiv Y_{n_1,\cdots,n_j}\left(\lambda_{d_1x_1,\cdots,d_jx_j}\right)=e^{-\lambda}\partial_{x_1}^{n_1}\cdots\partial_{x_j}^{n_j}e^{\lambda}, \quad (3)$$

it is named the multi-D Bell polynomials as follows:

$$\lambda_{d_1x_1,\cdots,d_jx_j}=\partial_{x_1}^{d_1}\cdots\partial_{x_j}^{d_j}\lambda,\quad \lambda_{0x_i}\equiv\lambda, d_1=0,\cdots,n_1\,;\cdots;d_j=0,\cdots,n_j, \quad (4)$$

and we have

$$\begin{aligned}
&Y_1(\lambda)=\lambda_x, Y_2(\lambda)=\lambda_{2x}+\lambda_x^2,\\
&Y_3(\lambda)=\lambda_{3x}+3\lambda_x\lambda_{2x}+\lambda_x^3,\cdots,\quad \lambda=\lambda(x,t),\\
&Y_{x,t}(\lambda)=\lambda_{x,t}+\lambda_x\lambda_t,\\
&Y_{2x,t}(\lambda)=\lambda_{2x,t}+\lambda_{2x}\lambda_t+2\lambda_{x,t}\lambda_x+\lambda_x^2\lambda_t,\cdots.
\end{aligned} \quad (5)$$

The multidimensional binary Bell polynomials can be written as

$$\begin{aligned}
&\Sigma_{n_1x_1,\cdots,n_jx_j}(\mu_1,\mu_2)\\
&=Y_{n_1,\cdots,n_j}(\lambda)\Big|_{\lambda_{d_1x_1,\cdots,d_jx_j}=\begin{cases}\mu_{1d_1x_1,\cdots,d_jx_j}, & d_1+d_2+\cdots+d_j, \text{ is odd},\\ \mu_{2d_1x_1,\cdots,d_jx_j}, & d_1+d_2+\cdots+d_j, \text{ is even}.\end{cases}}
\end{aligned} \quad (6)$$

We have the following conditions:

$$\begin{aligned}
&\Sigma_x(\mu_1)=\mu_{1x},\\
&\Sigma_{2x}(\mu_1,\mu_2)=\mu_{22x}+\mu_{1x}^2,\\
&\Sigma_{x,t}(\mu_1,\mu_2)=\mu_{2x,t}+\mu_{1x}\mu_{1t},\cdots.
\end{aligned} \quad (7)$$

Proposition 1. *Let $\mu_1=\ln(\Omega_1/\Omega_2), \mu_2=\ln(\Omega_1\Omega_2)$, then the relations between binary Bell polynomials and Hirota D-operator read*

$$\begin{aligned}
&\Sigma_{n_1x_1,\cdots,n_jx_j}(\mu_1,\mu_2)\Big|_{\mu_1=\ln(\Omega_1/\Omega_2),\mu_2=\ln(\Omega_1\Omega_2)}\\
&=(\Omega_1\Omega_2)^{-1}D_{x_1}^{n_1}\cdots D_{x_j}^{n_j}\Omega_1\Omega_2,
\end{aligned} \quad (8)$$

with Hirota operator

$$\begin{aligned}
\prod_{i=1}^{j}D_{x_i}^{n_i}g.\eta=\prod_{i=1}^{j}\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_{i'}}\right)^{n_i}\Omega_1(x_1,\cdots,x_j)\\
\cdot\Omega_2(x_{1'},\cdots,x_{j'})\Big|_{x_1=x_{1'},\cdots,x_j=x_{j'}}.
\end{aligned} \quad (9)$$

Proposition 2. *Take $\Xi(\gamma)=\sum_i\delta_i\mathfrak{P}_{d_1x_1,\cdots,d_jx_j}=0$ and $\mu_1=\ln(\Omega_1/\Omega_2), \mu_1=\ln(\Omega_1\Omega_2)$, we have*

$$\begin{cases}\displaystyle\sum_i\delta_{1i}Y_{n_1x_1,\cdots,n_jx_j}(\mu_1,\mu_2)=0,\\ \displaystyle\sum_i\delta_{1i}Y_{d_1x_1,\cdots,d_jx_j}(\mu_1,\mu_2)=0,\end{cases} \quad (10)$$

which need to satisfy

$$\mathfrak{R}(\gamma',\gamma)=\mathfrak{R}(\gamma')-\mathfrak{R}(\gamma)=\mathfrak{R}(\mu_2+\mu_1)-\mathfrak{R}(\mu_2-\mu_1)=0. \quad (11)$$

The generalized Bell polynomials $Y_{n_1x_1,\cdots,n_jx_j}(\xi)$ is as

$$
\begin{aligned}
&(\Omega_1\Omega_2)^{-1}D_{x_1}^{n_1}\cdots D_{x_j}^{n_j}\Omega_1\Omega_2 \\
&= \Sigma_{n_1x_1,\cdots,n_jx_j}(\mu_1,\mu_2)\Big|_{\mu_1=\ln(\Omega_1/\Omega_2),\,\mu_2=\ln(\Omega_1\Omega_2)} \\
&= \Sigma_{n_1x_1,\cdots,n_jx_j}(\mu_1,\mu_1+\gamma)\Big|_{\mu_1=\ln(\Omega_1/\Omega_2),\,\gamma=\ln(\Omega_1\Omega_2)} \\
&= \sum_{k_1}^{n_1}\cdots\sum_{k_j}^{n_j}\prod_{i=1}^{j}\binom{n_i}{k_i}\mathfrak{P}_{k_1x_1,\cdots,k_jx_j}(\gamma)Y_{(n_1-k_1)x_1,\cdots,(n_j-k_j)x_j}(\mu_1).
\end{aligned}
\tag{12}
$$

The Cole-Hopf relation is as follows:

$$
\begin{aligned}
&Y_{k_1x_1,\cdots,k_jx_j}(\mu_1=\ln(\varphi)) \\
&= \frac{\varphi_{n_1x_1,\cdots,n_jx_j}}{\varphi}, \\
&(\Omega_1\Omega_2)^{-1}D_{x_1}^{n_1}\cdots D_{x_j}^{n_j}\Omega_1\Omega_2\Big|_{\Omega_2=\exp\left(\frac{\gamma}{2}\right),\,\Omega_1/\Omega_2=\varphi} \\
&= \varphi^{-1}\sum_{k_1}^{n_1}\cdots\sum_{k_j}^{n_j}\prod_{l=1}^{j}\binom{n_l}{k_l}\mathfrak{P}_{k_1x_1,\cdots,k_lx_l}(\gamma)\varphi_{(n_1-k_1)x_1,\cdots,(n_d-k_l)x_l},
\end{aligned}
\tag{13}
$$

with

$$
\begin{aligned}
Y_t(\mu_1) &= \frac{\varphi_t}{\varphi}, \\
Y_{2x}(\mu_1,\beta) &= \gamma_{2x} + \frac{\varphi_{2x}}{\varphi}, \\
Y_{2x,y}(\mu_1,\mu_2) &= \frac{\gamma_{2x}\varphi_y}{\varphi} + \frac{2\gamma_{x,y}\varphi_x}{\varphi} + \frac{\varphi_{2x,y}}{\varphi}.
\end{aligned}
\tag{14}
$$

3. Bilinear Form Equation to the Equation in Liquid with Gas Bubble

To discover the linearizing representation, take the following form:

$$
\begin{aligned}
u &= c(t)\pi_{xx} + u_0, \\
\pi &= \pi(x,y,z,t).
\end{aligned}
\tag{15}
$$

Inserting Equation (15) into Equation (2), one obtains

$$
\begin{aligned}
\mathfrak{R}(\pi) &= \left(\frac{d}{dt}c(t)\right)\frac{\partial}{\partial x}\pi(x,y,z,t) + c(t)\frac{\partial^2}{\partial x\partial t}\pi(x,y,z,t) \\
&+ \phi_1(t)(c(t))^2\left(\frac{\partial}{\partial x}\pi(x,y,z,t)\right)^2 \\
&+ \phi_1(t)(c(t))^2\pi(x,y,z,t)\frac{\partial^2}{\partial x^2}\pi(x,y,z,t) \\
&+ \phi_2(t)c(t)\frac{\partial^4}{\partial x^4}\pi(x,y,z,t) + \phi_3(t)c(t)\frac{\partial^2}{\partial x^2}\pi(x,y,z,t) \\
&+ \phi_4(t)c(t)\frac{\partial^2}{\partial y^2}\pi(x,y,z,t) + \phi_5(t)c(t)\frac{\partial^2}{\partial z^2}\pi(x,y,z,t) = 0,
\end{aligned}
\tag{16}
$$

with

$$
c(t) = \frac{12\phi_2(t)}{\phi_1(t)}. \tag{17}
$$

The novel equation $\mathfrak{R}(\pi)$ is as

$$
\begin{aligned}
\mathfrak{R}(\pi) &= \mathfrak{P}_{x,t} + \phi_2(t)\left(\mathfrak{P}_{4x} + 3\mathfrak{P}_{2x}^2\right) + \phi_3(t)\mathfrak{P}_{2x} \\
&+ \phi_4(t)\mathfrak{P}_{2y} + \phi_5(t)\mathfrak{P}_{2z} \\
&= 0.
\end{aligned}
\tag{18}
$$

Applying a change of dependent variable,

$$
\pi = \ln(g) \Leftrightarrow u = \frac{12\phi_2(t)}{\phi_1(t)}\ln(g)_{xx}. \tag{19}
$$

Theorem 3. *With the following relations:*

$$
\pi = \ln(g) \Leftrightarrow u = \frac{12\phi_2(t)}{\phi_1(t)}\ln(g)_{xx}, \tag{20}
$$

into Equation (2), the $(3+1)$-D VC NLW equation can be stated as follows:

$$
\begin{aligned}
\mathfrak{R}(g) &= (gg_{xt} - g_xg_t) + \phi_2(t)\left(gg_{4x} - 4g_xg_{3x} + 3g_{xx}^2\right) \\
&+ \phi_3(t)\left(gg_{xx} - g_x^2\right) \\
&\phi_4(t)\left(gg_{yy} - g_y^2\right) + \phi_5(t)\left(gg_{zz} - g_z^2\right) \\
&= \frac{1}{2}\Big(D_xD_t + \phi_2(t)D_x^4 + \phi_3(t)D_x^2 + \phi_4(t)D_y^2 \\
&+ \phi_5(t)D_z^2\Big)g\cdot g = 0,
\end{aligned}
\tag{21}
$$

where $g = g(x,y,z,t)$ and $\pi = \pi(x,y,z,t)$.

4. Damped Oscillation Solutions for Generalized CV NLW Equation

The two subsections including damped sinusoidal oscillations wave solutions and combined damped sinusoidal oscillation wave solutions are investigated in the following subsections.

4.1. Damped Oscillation Wave Solutions. Here, we utilize to formulate the new exact solutions to the $(3+1)$-dimensional generalized CV NLW equation. Consider the below function for studying the damped oscillation wave solutions as

$$
\begin{aligned}
g &= e^{a_1}(\sigma_4(t)\cos(a_2) + \sigma_5(t)\sin(a_3)), \\
a_l &= \alpha_lx + \beta_ly + \delta_lz + \sigma_l(t), \quad l = 1,2,3.
\end{aligned}
\tag{22}
$$

Afterwards, the values $\alpha_l, \beta_l, \delta_l, \sigma_l(t), (l = 1, 2, 3)$ will be found. By making use of Equation (22) into (21) and taking the coefficients, each power of cos (a_2) and sin (a_2) to zero yields a system of equations (algebraic) (these are not collected here for the minimalist) for $\alpha_l, \beta_l, \delta_l, \sigma_l(t), (l = 1, 2, 3)$. These algebraic equations by using the emblematic computation software like Maple give the solutions in the following with using $u = (12\phi_2/\phi_1)(\ln g)_{xx}$ as follows:

$$\begin{aligned}
&4\,\alpha_2{}^4\phi_2 s_4{}^2 + 4\,\alpha_3{}^4\phi_2 s_5{}^2 - \alpha_2{}^2\phi_3 s_4{}^2 - \alpha_3{}^2\phi_3 s_5{}^2 - \beta_2{}^2\phi_4 s_4{}^2 \\
&\quad - \beta_3{}^2\phi_4 s_5{}^2 - \delta_2{}^2\phi_5 s_4{}^2 - \delta_3{}^2\phi_5 s_5{}^2 - S_2\alpha_2 s_4{}^2 - S_3\alpha_3 s_5{}^2 = 0, \\
&-S_4\alpha_3 s_5 + S_5\alpha_3 s_4 = 0, \\
&\alpha_2{}^4\phi_2 s_4 s_5 + 6\,\alpha_2{}^2\alpha_3{}^2\phi_2 s_4 s_5 + \alpha_3{}^4\phi_2 s_4 s_5 - \alpha_2{}^2\phi_3 s_4 s_5 \\
&\quad - \alpha_3{}^2\phi_3 s_4 s_5 - \beta_2{}^2\phi_4 s_4 s_5 - \beta_3{}^2\phi_4 s_4 s_5 - \delta_2{}^2\phi_5 s_4 s_5 \\
&\quad - \delta_3{}^2\phi_5 s_4 s_5 - S_2\alpha_2 s_4 s_5 - S_3\alpha_3 s_4 s_5 = 0, \\
&-4\,\alpha_2{}^3\alpha_3\phi_2 s_4 s_5 - 4\,\alpha_2\alpha_3{}^3\phi_2 s_4 s_5 + 2\,\alpha_2\alpha_3\phi_3 s_4 s_5 + 2\,\beta_2\beta_3\phi_4 s_4 s_5 \\
&\quad + 2\,\delta_2\delta_3\phi_5 s_4 s_5 + S_2\alpha_3 s_4 s_5 + S_3\alpha_2 s_4 s_5 = 0, \\
&-S_4\alpha_2 s_5 + S_5\alpha_2 s_4 = 0,
\end{aligned} \tag{23}$$

where $s_l = \sigma_i(t)$ and $S_l = (d/dt)\sigma_l(t), l = 1, \cdots, 5$ and the obtained solutions are as follows:

4.1.1. Set I Solutions

$$\begin{aligned}
\sigma_5(t) &= \frac{\sqrt{\left(3\,\alpha_3{}^2\phi_2(\alpha_2-\alpha_3)^2(\alpha_2+\alpha_3)^2 + A_1\right)\left(3\,\alpha_2{}^2\phi_2(\alpha_2-\alpha_3)^2(\alpha_2+\alpha_3)^2 + A_1\right)}\,\sigma_4(t)}{3\,\alpha_3{}^2\phi_2(\alpha_2-\alpha_3)^2(\alpha_2+\alpha_3)^2 + A_1}, \\
\sigma_2(t) &= \frac{\left(\alpha_2\left(\alpha_2{}^2\phi_2 + 3\,\alpha_3{}^2\phi_2 - \phi_3\right)\left(\alpha_2{}^2 - \alpha_3{}^2\right) + A_2\right)t}{\alpha_2{}^2 - \alpha_3{}^2} + C_1, \\
\sigma_3(t) &= \frac{\left(\alpha_3\left(3\,\alpha_2{}^2\phi_2 + \alpha_3{}^2\phi_2 - \phi_3\right)\left(\alpha_2{}^2 - \alpha_3{}^2\right) + A_3\right)t}{\alpha_2{}^2 - \alpha_3{}^2} + C_2, \\
A_1 &= \phi_4(\alpha_2\beta_3 - \alpha_3\beta_2)^2 + \phi_5(\alpha_2\delta_3 - \alpha_3\delta_2)^2, \\
A_2 &= -\phi_4\left(\alpha_2\beta_2{}^2 + \alpha_2\beta_3{}^2 - 2\,\alpha_3\beta_2\beta_3\right) - \phi_5\left(\alpha_2\delta_2{}^2 + \alpha_2\delta_3{}^2 - 2\,\alpha_3\delta_2\delta_3\right), \\
A_3 &= -\phi_4\left(2\,\alpha_2\beta_2\beta_3 - \alpha_3\beta_2{}^2 - \alpha_3\beta_3{}^2\right) - \phi_5\left(2\,\alpha_2\delta_2\delta_3 - \alpha_3\delta_2{}^2 - \alpha_3\delta_3{}^2\right),
\end{aligned} \tag{24}$$

where $\phi_l = \phi_l(t), l = 1, \cdots, 5$ and $\sigma_1(t), \sigma_4(t)$ free functions of t and $\alpha_l, \beta_l, \delta_l, (l = 1, 2, 3)$, are free values. It concludes along with the bilinear equation; the exact solution will be as

$$\begin{aligned}
\Sigma_1 &= \frac{12\phi_2}{\phi_1}(\ln g(x,y,z,t))_{xx} \\
&= 2\,\frac{\left(\left(\partial^2/\partial x^2\right)g(x,y,z,t)\right)g(x,y,z,t) - \left((\partial/\partial x)g(x,y,z,t)\right)^2}{(g(x,y,z,t))^2},
\end{aligned} \tag{25}$$

$$\begin{aligned}
g(x,y,z,t) &= e^{\alpha_1 x + \beta_1 y + \delta_1 z + \sigma_1(t)}(\sigma_4(t)\cos \\
&\quad \cdot(\alpha_2 x + \beta_2 y + \delta_2 z + \sigma_2(t)) \\
&\quad + \sigma_5(t)\sin(\alpha_3 x + \beta_3 y + \delta_3 z + \sigma_3(t))).
\end{aligned} \tag{26}$$

Figures 1–3, respectively, show the analysis of treatment of periodic wave solution with graphs of Σ with the below-selected parameters:

$$\begin{aligned}
&C_1 = C_2 = 1, \alpha_1 = \beta_2 = \delta_1 = 0.2, \alpha_2 = \beta_1 = 0.1, \\
&\alpha_3 = \beta_3 = \delta_2 = 0.3, \delta_3 = 0.4, \\
&\sigma_1(t) = \sigma_4(t) = \phi_1 = \phi_2 = \phi_3 = \phi_4 = t, \phi_5 = 2t, z = 1, y = 2, \\
&C_1 = C_2 = \alpha_1 = \beta_2 = \delta_1 = \alpha_2 = \beta_1 = \beta_3 = \delta_2 \\
&\quad = \delta_3 = 1, \quad \alpha_3 = 2, \sigma_1(t) = 1, \sigma_4(t) = 2, \\
&\phi_1 = \cos(t), \phi_2 = \cos(2t), \phi_3 = \cos(2t+1), \\
&\phi_4 = \cos(2t+2), \phi_5 = \cos(2t+3), z = 1, y = 1, \\
&C_1 = C_2 = \alpha_1 = \beta_2 = \delta_1 = \alpha_2 = \beta_1 = \beta_3 = \delta_2 = \delta_3 = 1, \\
&\alpha_3 = 2, \sigma_1(t) = 1, \sigma_4(t) = 2, \\
&\phi_1 = 1, \phi_2 = 1, \phi_3 = 1, \phi_4 = 1, \phi_5 = 1, z = 1, y = -1,
\end{aligned} \tag{27}$$

in Equation (25).

4.1.2. Set II Solutions

$$\beta_3 = \frac{\alpha_3\beta_2}{\alpha_2},$$

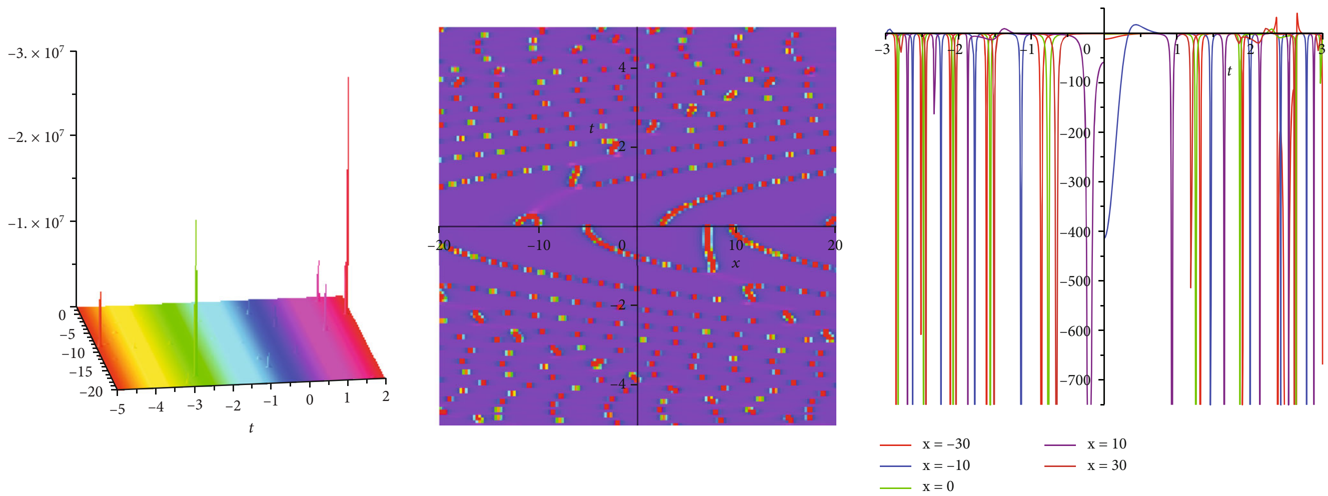

Figure 1: Plot of periodic wave solution (25) (Σ_1) (3D plot, density plot, and 2D plot t), respectively.

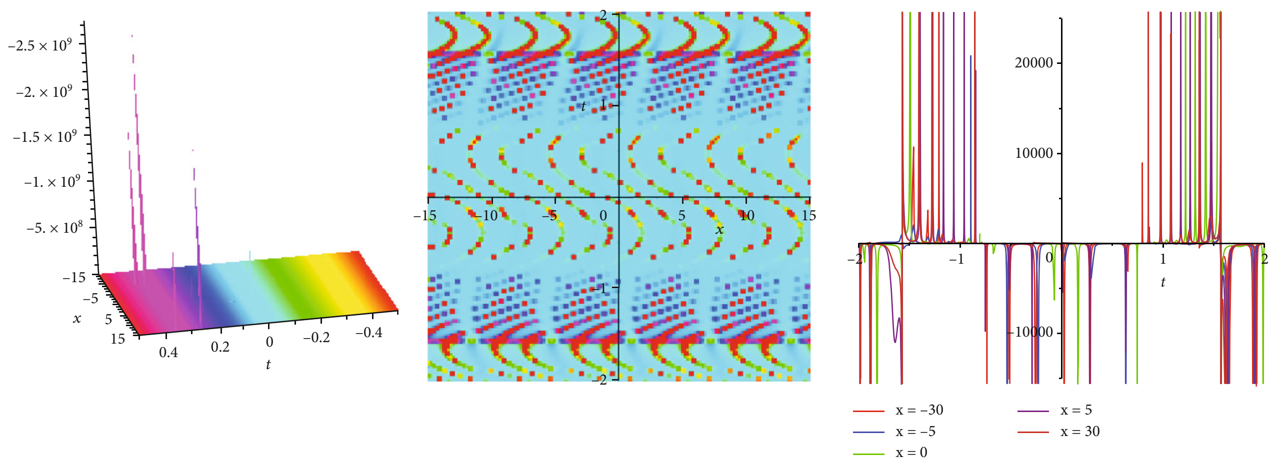

Figure 2: Plot of periodic wave solution (25) (Σ_1) (3D plot, density plot, and 2D plot t), respectively.

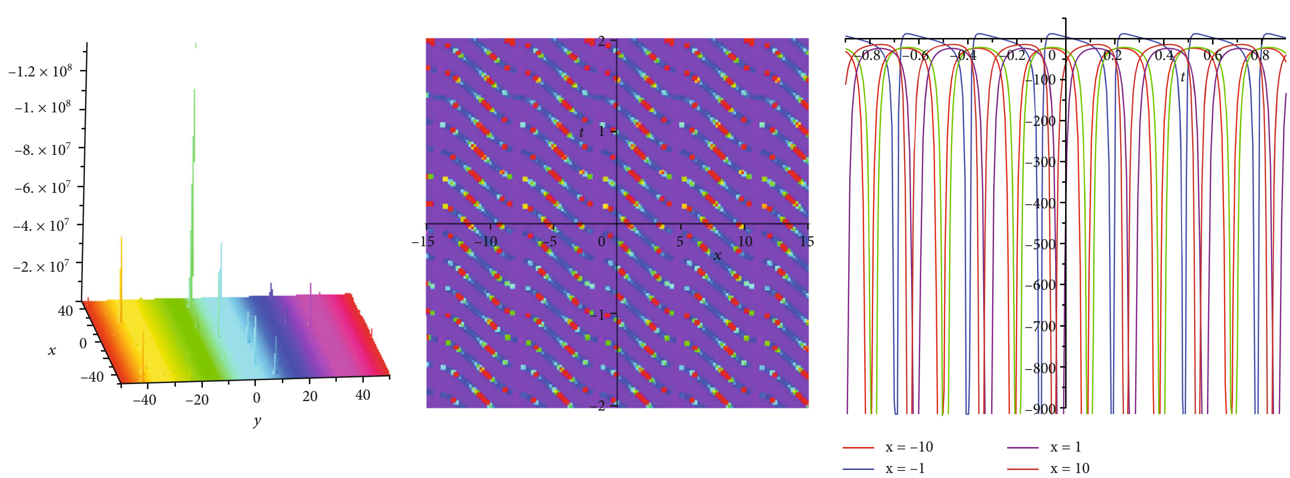

Figure 3: Plot of periodic wave solution (25) (Σ_1) (3D plot, density plot, and 2D plot t), respectively.

$$\sigma_5(t)=\frac{\sqrt{3\alpha_2^2\phi_2(\alpha_2^2-\alpha_3^2)^2+\phi_5(\alpha_2\delta_3-\alpha_3\delta_2)^2}\sigma_4(t)}{\sqrt{3\alpha_3^2\phi_2(\alpha_2^2-\alpha_3^2)^2+\phi_5(\alpha_2\delta_3-\alpha_3\delta_2)^2}},$$
$$\sigma_2(t)=\frac{\left(\alpha_2^2\phi_2\left(\alpha_2^2+3\alpha_3^2\right)\left(\alpha_2^2-\alpha_3^2\right)+A_1\right)t}{\alpha_2(\alpha_2^2-\alpha_3^2)}+C_1,$$
$$\sigma_3(t)=\frac{\left(\alpha_2^2\alpha_3\phi_2\left(3\alpha_2^2+\alpha_3^2\right)\left(\alpha_2^2-\alpha_3^2\right)+A_2\right)t}{\alpha_2^2(\alpha_2^2-\alpha_3^2)}+C_2,$$
$$A_1=-\left(\alpha_2^2-\alpha_3^2\right)\left(\alpha_2^2\phi_3+\beta_2^2\phi_4\right)-\alpha_2\phi_5\left(\alpha_2\delta_2^2+\alpha_2\delta_3^2-2\alpha_3\delta_2\delta_3\right),$$
$$A_2=-\alpha_3\left(\alpha_2^2\phi_3+\beta_2^2\phi_4\right)\left(\alpha_2^2-\alpha_3^2\right)-\alpha_2^2\phi_5\left(2\alpha_2\delta_2\delta_3-\alpha_3\delta_2^2-\alpha_3\delta_3^2\right), \tag{28}$$

where $\sigma_1(t), \sigma_4(t)$ free functions of t and $\alpha_l, \beta_l, \delta_l, (l=1,2,3)$, are free values. It concludes along with the bilinear equation; the exact solution will be as

$$\Sigma_2=\frac{12\phi_2}{\phi_1}(\ln g(x,y,z,t))_{xx},$$
$$g(x,y,z,t)=e^{\alpha_1x+\beta_1y+\delta_1z+\sigma_1(t)}\cdot\left(\sigma_4(t)\cos\left(\alpha_2x+\beta_2y+\delta_2z+\sigma_2(t)\right)+\sigma_5(t)\sin\left(\alpha_3x+\frac{\alpha_3\beta_2y}{\alpha_2}+\delta_3z+\sigma_3(t)\right)\right). \tag{29}$$

4.1.3. Set III Solutions

$$\sigma_5(t)=\pm\frac{\sqrt{\left(3\alpha_3^4\phi_2+\delta_2^2\phi_5\right)\phi_5}\delta_2\sigma_4(t)}{3\alpha_3^4\phi_2+\delta_2^2\phi_5},$$
$$\sigma_2(t)=-2\frac{\delta_3\delta_2\phi_5t}{\alpha_3}+C_1,$$
$$\sigma_3(t)=\frac{\left(\alpha_3^4\phi_2-\alpha_3^2\phi_3-\beta_3^2\phi_4-\delta_2^2\phi_5-\delta_3^2\phi_5\right)t}{\alpha_3}+C_2,$$
$$\alpha_2=\beta_2=0, \tag{30}$$

where $\sigma_1(t), \sigma_4(t)$ free functions of t and $\alpha_l, \beta_l, \delta_l, (l=1,2,3)$, are free values. It concludes along with the bilinear equation; the exact solution will be as

$$\Sigma_3=\frac{12\phi_2}{\phi_1}(\ln g(x,y,z,t))_{xx},$$
$$g(x,y,z,t)=e^{\alpha_1x+\beta_1y+\delta_1z+\sigma_1(t)}\left(\sigma_4(t)\cos\left(\delta_2z+\sigma_2(t)\right)+\sigma_5(t)\sin\left(\alpha_3x+\beta_3y+\delta_3z+\sigma_3(t)\right)\right). \tag{31}$$

4.1.4. Set IV Solutions

$$\sigma_2(t)=\frac{\left(4\alpha_2^4\phi_2-\alpha_2^2\phi_3-\beta_3^2\phi_4-\delta_3^2\phi_5\right)t}{\alpha_2}+C_1,$$
$$\sigma_3(t)=\frac{\left(4\alpha_2^4\phi_2-\alpha_2^2\phi_3-\beta_3^2\phi_4-\delta_3^2\phi_5\right)t}{\alpha_2}+C_2,$$
$$\sigma_5(t)=C_3\sigma_4(t), \tag{32}$$

where $\sigma_1(t), \sigma_4(t)$ free functions of t and $\alpha_l, \beta_l, \delta_l, (l=1,2,3)$, are free values. It concludes along with the bilinear equation; the exact solution will be as

$$\Sigma_4=\frac{12\phi_2}{\phi_1}(\ln g(x,y,z,t))_{xx},$$
$$g(x,y,z,t)=e^{\alpha_1x+\beta_1y+\delta_1z+\sigma_1(t)}\left(\sigma_4(t)\cdot\cos\left(\alpha_2x+\beta_3y+\delta_3z+\sigma_2(t)\right)+\sigma_5(t)\sin\left(\alpha_2x+\beta_3y+\delta_3z+\sigma_3(t)\right)\right). \tag{33}$$

4.1.5. Set V Solutions

$$\sigma_2(t)=\int-\frac{-(\sigma_5(t))^2\alpha_3^4\phi_2\left(3(\sigma_4(t))^2+(\sigma_5(t))^2\right)+\alpha_3^2\phi_3(\sigma_4(t))^2(\sigma_5(t))^2+(\sigma_4(t))^4\left(\beta_2^2\phi_4+\delta_2^2\phi_5\right)}{\alpha_3(\sigma_4(t))^3\sigma_5(t)}dt+C_1,$$
$$\sigma_3(t)=\int-\frac{-(\sigma_5(t))^2\alpha_3^4\phi_2\left((\sigma_4(t))^2+3(\sigma_5(t))^2\right)+\alpha_3^2\phi_3(\sigma_4(t))^2(\sigma_5(t))^2+(\sigma_4(t))^4\left(\beta_2^2\phi_4+\delta_2^2\phi_5\right)}{(\sigma_4(t))^2(\sigma_5(t))^2\alpha_3}dt+C_2,$$
$$\sigma_5(t)=C_3\sigma_4(t),$$
$$\alpha_2=C_3\alpha_3,$$
$$\beta_3=\frac{\beta_2}{C_3},$$
$$\delta_3=\frac{\delta_2}{C_3}, \tag{34}$$

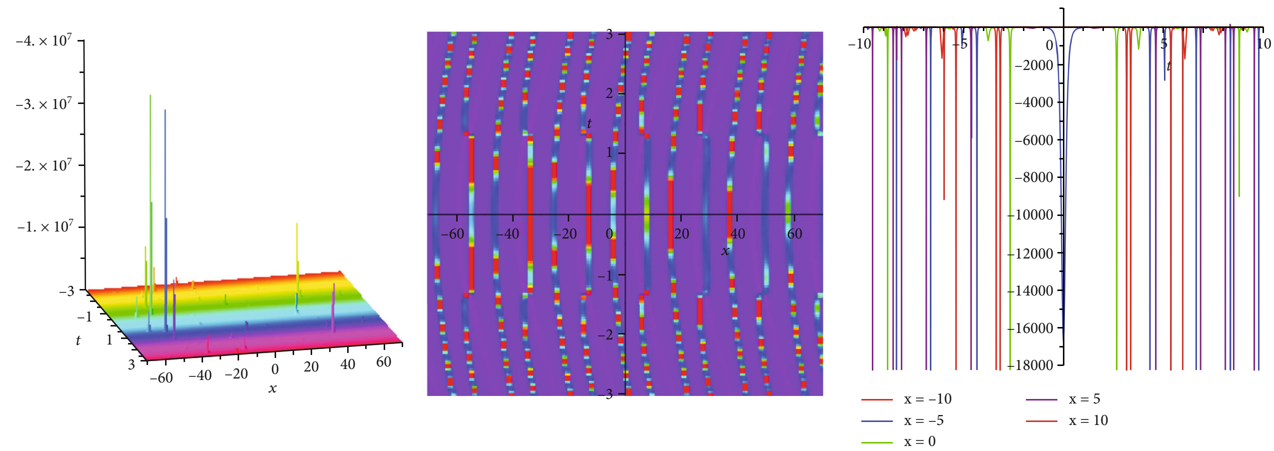

FIGURE 4: Plot of periodic wave solution (35) (Σ_5) (3D plot, density plot, and 2D plot t), respectively.

where $\sigma_1(t), \sigma_4(t)$ free functions of t and $\alpha_l, \beta_l, \delta_l, (l = 1, 2, 3)$, are free values. It concludes along with the bilinear equation; the exact solution will be as

$$\begin{aligned}\Sigma_5 &= \frac{12\phi_2}{\phi_1}(\ln g(x, y, z, t))_{xx},\\ g(x, y, z, t) &= e^{\alpha_1 x+\beta_1 y+\delta_1 z+\sigma_1(t)}\\ &\cdot \left(\sigma_4(t)\cos\left(C_3\alpha_3 x + \beta_2 y + \delta_2 z + \sigma_2(t)\right) + \sigma_5(t)\right.\\ &\left.\cdot \sin\left(\alpha_3 x + \frac{\beta_2 y}{C_3} + \frac{\delta_2 z}{C_3} + \sigma_3(t)\right)\right).\end{aligned} \tag{35}$$

Figures 4–6, respectively, show the analysis of treatment of periodic wave solution with graphs of Σ with the below-selected parameters:

$$\begin{aligned}&C_1 = C_2 = 1, C_3 = 2, \alpha_1 = \beta_2 = \delta_1 = 0.2, \beta_1 = 0.1,\\ &\alpha_3 = \beta_3 = \delta_2 = 0.3, \delta_3 = 0.4,\\ &\sigma_1(t) = \sigma_4(t) = \phi_1 = \phi_2 = \phi_3 = \phi_4 = t, \phi_5 = 2t, z = 1, y = 2,\\ &C_1 = C_2 = 1, C_3 = 2, \alpha_1 = \beta_2 = \delta_1 = 0.2, \beta_1 = 0.1,\\ &\alpha_3 = \beta_3 = \delta_2 = 0.3, \delta_3 = 0.4, \sigma_1(t) = t, \sigma_4(t) = t\cos(t),\\ &\phi_1 = t + 1, \phi_2 = \sin(t), \phi_3 = t + t^2, \phi_4 = t - t^3,\\ &\phi_5 = 2t + 1, z = 1, y = 2,\\ &C_1 = C_2 = 1, C_3 = 2, \alpha_1 = \beta_2 = \delta_1 = 0.2, \beta_1 = 0.1,\\ &\alpha_3 = \beta_3 = \delta_2 = 0.3, \delta_3 = 0.4, \sigma_1(t) = t, \sigma_4(t) = \sin\left(t^2\right),\\ &\phi_1 = 1, \phi_2 = 2, \phi_3 = 3, \phi_4 = 3, \phi_5 = 2, z = 1, y = 1,\end{aligned} \tag{36}$$

in Equation (35).

4.2. Combined Damped Oscillation Wave Solutions. Here, we utilize to formulate the new exact solutions to the $(3+1)$-dimensional generalized CV NLW equation. Consider the below function for studying the combined damped oscillation wave solutions as

$$\begin{aligned}g &= e^{a_1}(\sigma_5(t)\cos(a_2) + \sigma_6(t)\sin(a_2))\\ &\quad + e^{a_3}(\sigma_7(t)\cos(a_4) + \sigma_8(t)\sin(a_5)),\\ a_l &= \alpha_l x + \beta_l y + \delta_l z + \sigma_l(t), \quad l = 1, 2, 3, 4.\end{aligned} \tag{37}$$

Afterwards, the values $\alpha_l, \beta_l, \delta_l, \sigma_l(t), (l = 1, 2, 3, 4)$ will be found. By making use of Equation (37) into (21) and taking the coefficients, each power of $\cos(a_2)$ and $\sin(a_2)$, $\cos(a_4)$ and $\sin(a_4)$ to zero yields a system of equations (algebraic) (these are not collected here for the minimalist) for $\alpha_l, \beta_l, \delta_l, \sigma_l(t), (l = 1, 2, 3, 4)$. These algebraic equations by using the emblematic computation software like Maple give the solutions in the following with using $u = (12\phi_2/\phi_1)(\ln g)_{xx}$ as follows:

4.2.1. Set I Solutions

$$\begin{aligned}&\sigma_7(t) = 0,\\ &\sigma_4(t) = C_3,\\ &\sigma_2(t) = -\arctan\left(\frac{\sigma_5(t)}{\sigma_6(t)}\right)\\ &\qquad + 2\,\frac{\delta_2\phi_5(\beta_1\delta_2 - \beta_2\delta_1 + \beta_2\delta_3 - \beta_3\delta_2)t}{\alpha_1\beta_2 - \alpha_3\beta_2} + C_1,\\ &\sigma_3(t) = \int M(t)dt + C_2,\\ &\alpha_2 = \alpha_4 = \beta_4 = \delta_4 = 0,\\ &\phi_4 = -\frac{\delta_2{}^2\phi_5}{\beta_2{}^2},\end{aligned}$$

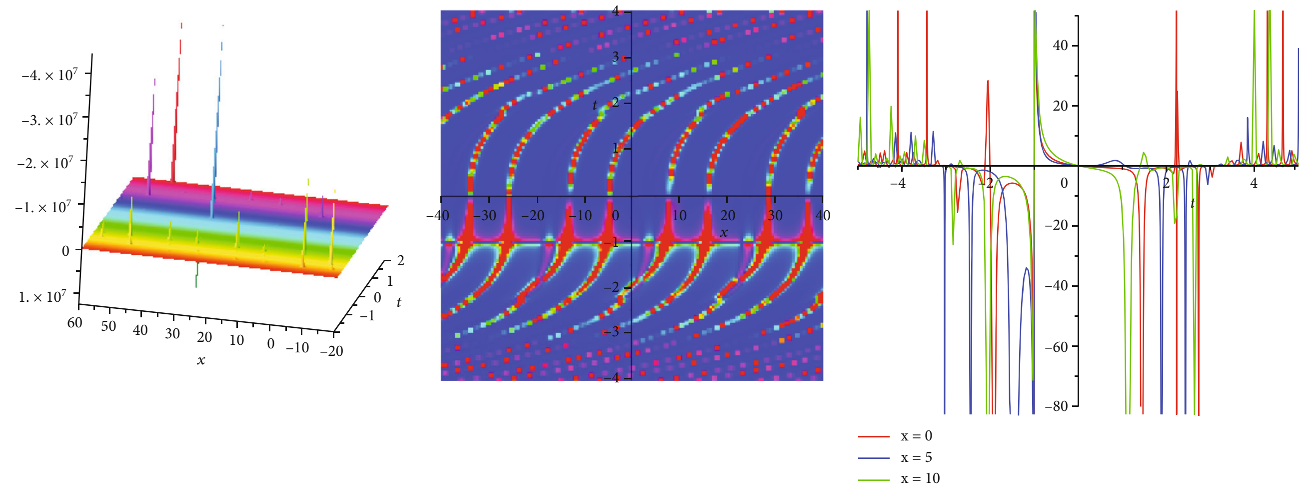

Figure 5: Plot of periodic wave solution (35) (Σ_5) (3D plot, density plot, and 2D plot t), respectively.

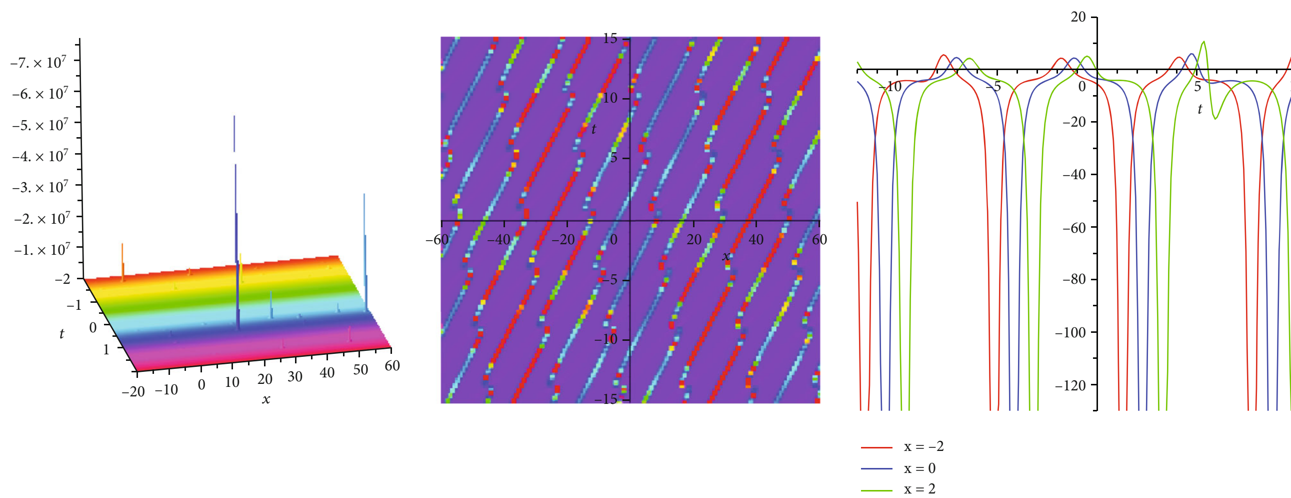

Figure 6: Plot of periodic wave solution (35) (Σ_5) (3D plot, density plot, and 2D plot t), respectively.

$$M(t)=\frac{-\sigma_8(t)(A_5+A_6-A_7)+A_1+\sigma_8(t)(A_2+A_3)-A_4}{\sigma_8(t)\beta_2^2\left((\sigma_5(t))^2+(\sigma_6(t))^2\right)(\alpha_1-\alpha_3)},$$

$$A_1=\sigma_8(t)\left(\frac{d}{dt}\sigma_1(t)\right)\beta_2^2\left((\sigma_5(t))^2+(\sigma_6(t))^2\right)(\alpha_1-\alpha_3)+\sigma_8(t)\beta_2^2(\alpha_1-\alpha_3)\cdot\left(\left(\frac{d}{dt}\sigma_6(t)\right)\sigma_6(t)+\sigma_5(t)\frac{d}{dt}\sigma_5(t)\right),$$

$$A_2=(\sigma_6(t))^2\left(\alpha_1^4\beta_2^2\phi_2+\alpha_3^4\beta_2^2\phi_2+\alpha_1^2\beta_2^2\phi_3+\alpha_3^2\beta_2^2\phi_3-\beta_1^2\delta_2^2\phi_5+\beta_2^2\delta_1^2\phi_5+\beta_2^2\delta_3^2\phi_5-\beta_3^2\delta_2^2\phi_5\right),$$

$$A_3=(\sigma_5(t))^2\left(\alpha_1^4\alpha_3^4\beta_2^4\phi_2^2+\alpha_3^2\beta_2^2\phi_3+1\right),$$

$$A_4=\left(\frac{d}{dt}\sigma_8(t)\right)\beta_2^2\left((\sigma_5(t))^2+(\sigma_6(t))^2\right)(\alpha_1-\alpha_3),$$

$$A_5=2\alpha_1\alpha_3\beta_2^2\phi_2\left(2\alpha_1^2-3\alpha_1\alpha_3+2\alpha_3^2\right)\cdot\left((\sigma_5(t))^2+(\sigma_6(t))^2\right),$$

$$A_6=2(\sigma_6(t))^2\left(\alpha_1\alpha_3\beta_2^2\phi_3-\beta_1\beta_3\delta_2^2\phi_5+\beta_2^2\delta_1\delta_3\phi_5\right),$$

$$A_7=(\sigma_5(t))^2\left(\alpha_1^2\beta_2^2\phi_3-2\alpha_1\alpha_3\beta_2^2\phi_3-\beta_1^2\delta_2^2\phi_5+2\beta_1\beta_3\delta_2^2\phi_5+\beta_2^2\delta_1^2\phi_5-2\beta_2^2\delta_1\delta_3\phi_5+\beta_2^2\delta_3^2\phi_5-\beta_3^2\delta_2^2\phi_5\right), \tag{38}$$

where $\sigma_1(t),\sigma_4(t)$ free functions of t and $\alpha_l,\beta_l,\delta_l,(l=1,2,3)$, are free values. It concludes along with the bilinear equation; the exact solution will be as

$$\Sigma_1=\frac{12\phi_2}{\phi_1}(\ln g)_{xx},$$

$$g = e^{\alpha_1 x+\beta_1 y+\delta_1 z+\sigma_1(t)}(\sigma_5(t)\cos(\beta_2 y+\delta_2 z+\sigma_2(t)) + \sigma_6(t)\sin(\beta_2 y+\delta_2 z+\sigma_2(t))) + e^{\alpha_3 x+\beta_3 y+\delta_3 z+\sigma_3(t)}(\sigma_7(t)\cos(\sigma_4(t)) + \sigma_8(t)\sin(\sigma_4(t))). \tag{39}$$

Set I-I:

$$\begin{aligned}
&\sigma_6(t) = 0,\\
&\sigma_8(t) = -\tan(-\sigma_4(t)+C_3)\sigma_7(t),\\
&\sigma_2(t) = \frac{2\delta_2\phi_5(\beta_1\delta_2-\beta_2\delta_1+\beta_2\delta_3-\beta_3\delta_2)t}{\beta_2(\alpha_1-\alpha_3)} + C_1,\\
&\sigma_3(t) = \int M(t)dt + C_2,\\
&\quad \alpha_2 = \alpha_4 = \beta_4 = \delta_4 = 0,\\
&\quad \phi_4 = -\frac{\delta_2{}^2\phi_5}{\beta_2{}^2},\\
&M(t) = -\frac{A_1+A_2}{\sigma_7(t)\sigma_5(t)\beta_2{}^2(\alpha_1-\alpha_3)} + C_2,\\
&\quad A_1 = -\sigma_5(t)\sigma_7(t)\big(\beta_2{}^2\phi_2(\alpha_1-\alpha_3)^4 + \beta_2{}^2\phi_3(\alpha_1-\alpha_3)^2\\
&\qquad - \phi_5(\beta_1\delta_2-\beta_2\delta_1+\beta_2\delta_3-\beta_3\delta_2)\\
&\qquad \cdot(\beta_1\delta_2+\beta_2\delta_1-\beta_2\delta_3-\beta_3\delta_2)\big),\\
&\quad A_2 = \beta_2{}^2(\alpha_1-\alpha_3)\left(\sigma_8(t)\left(\frac{d}{dt}\sigma_4(t)\right)\sigma_5(t)\right.\\
&\qquad - \sigma_5(t)\sigma_7(t)\frac{d}{dt}\sigma_1(t) + \sigma_5(t)\frac{d}{dt}\sigma_7(t)\\
&\qquad \left. - \sigma_7(t)\frac{d}{dt}\sigma_5(t)\right),
\end{aligned} \tag{40}$$

where $\sigma_1(t), \sigma_4(t)$ free functions of t and $\alpha_l, \beta_l, \delta_l, (l = 1, 2, 3)$, are free values. It concludes along with the bilinear equation; the exact solution will be as

$$\begin{aligned}
&\Sigma_2 = \frac{12\phi_2}{\phi_1}(\ln g)_{xx},\\
&g = e^{\alpha_1 x+\beta_1 y+\delta_1 z+\sigma_1(t)}(\sigma_5(t)\cos(\beta_2 y+\delta_2 z+\sigma_2(t))\\
&\quad + \sigma_6(t)\sin(\beta_2 y+\delta_2 z+\sigma_2(t)))\\
&\quad + e^{\alpha_3 x+\beta_3 y+\delta_3 z+\sigma_3(t)}(\sigma_7(t)\cos(\sigma_4(t))\\
&\quad + \sigma_8(t)\sin(\sigma_4(t))).
\end{aligned} \tag{41}$$

Set I-II:

$$\begin{aligned}
&\sigma_8(t) = -\tan(-\sigma_4(t)+C_3)\sigma_7(t),\\
&\sigma_2(t) = -\arctan\left(\frac{\sigma_5(t)}{\sigma_6(t)}\right)\\
&\qquad + 2\,\frac{\delta_2\phi_5(\beta_1\delta_2-\beta_2\delta_1+\beta_2\delta_3-\beta_3\delta_2)t}{\alpha_1\beta_2-\alpha_3\beta_2} + C_1,\\
&\sigma_3(t) = \int M(t)dt + C_2,\\
&\quad \alpha_2 = \alpha_4 = \beta_4 = \delta_4 = 0,\\
&\quad \phi_4 = -\frac{\delta_2{}^2\phi_5}{\beta_2{}^2},\\
&M(t) = -\frac{A_1+A_2+A_3}{\sigma_7(t)\beta_2{}^2\left((\sigma_5(t))^2+(\sigma_6(t))^2\right)(\alpha_1-\alpha_3)} + C_2,\\
&\quad A_1 = 2\sigma_7(t)\left((\sigma_5(t))^2+(\sigma_6(t))^2\right)\big(2\alpha_1{}^3\alpha_3\beta_2{}^2\phi_2\\
&\qquad - 3\alpha_1{}^2\alpha_3{}^2\beta_2{}^2\phi_2 + 2\alpha_1\alpha_3{}^3\beta_2{}^2\phi_2 + \alpha_1\alpha_3\beta_2{}^2\phi_3\\
&\qquad - \beta_1\beta_3\delta_2{}^2\phi_5 + \beta_2{}^2\delta_1\delta_3\phi_5\big),\\
&\quad A_2 = \beta_2{}^2(\alpha_1-\alpha_3)\Bigg(\left((\sigma_5(t))^2+(\sigma_6(t))^2\right)\\
&\qquad \cdot\left(\sigma_8(t)\frac{d}{dt}\sigma_4(t) - \sigma_7(t)\frac{d}{dt}\sigma_1(t)\right)\\
&\qquad + (\sigma_6(t))^2\frac{d}{dt}\sigma_7(t) - \sigma_6(t)\left(\frac{d}{dt}\sigma_6(t)\right)\sigma_7(t)\\
&\qquad + (\sigma_5(t))^2\frac{d}{dt}\sigma_7(t) - \sigma_5(t)\sigma_7(t)\frac{d}{dt}\sigma_5(t)\Bigg),\\
&\quad A_3 = -\sigma_7(t)\left((\sigma_5(t))^2+(\sigma_6(t))^2\right)\big(\alpha_1{}^4\beta_2{}^2\phi_2 + \alpha_3{}^4\beta_2{}^2\phi_2\\
&\qquad + \alpha_1{}^2\beta_2{}^2\phi_3 + \alpha_3{}^2\beta_2{}^2\phi_3 - \beta_1{}^2\delta_2{}^2\phi_5 + \beta_2{}^2\delta_1{}^2\phi_5\\
&\qquad + \beta_2{}^2\delta_3{}^2\phi_5 - \beta_3{}^2\delta_2{}^2\phi_5\big),
\end{aligned} \tag{42}$$

where $\sigma_1(t), \sigma_4(t)$ free functions of t and $\alpha_l, \beta_l, \delta_l, (l = 1, 2, 3, 4)$, are free values. It concludes along with the bilinear equation; the exact solution will be as

$$\Sigma_3 = \frac{12\phi_2}{\phi_1}(\ln g)_{xx},\; g = e^{\alpha_1 x+\beta_1 y+\delta_1 z+\sigma_1(t)}(\sigma_5(t)\cos(\beta_2 y+\delta_2 z + \sigma_2(t)) + \sigma_6(t)\sin(\beta_2 y+\delta_2 z+\sigma_2(t))) + e^{\alpha_3 x+\beta_3 y+\delta_3 z+\sigma_3(t)}(\sigma_7(t)\cos(\sigma_4(t)) + \sigma_8(t)\sin(\sigma_4(t))). \tag{43}$$

4.2.2. Set II Solutions

$$\begin{aligned}
&\sigma_4(t) = C_3,\\
&\sigma_7(t) = 0,\\
&\sigma_2(t) = -\arctan\left(\frac{\sigma_5(t)}{\sigma_6(t)}\right) + C_1,
\end{aligned}$$

$$\alpha_2=\beta_2=\delta_2=\delta_4=0,$$
$$\sigma_3(t)=\int\frac{A_4+A_1+A_2+A_3}{\sigma_8(t)\left((\sigma_5(t))^2+(\sigma_6(t))^2\right)(\alpha_1-\alpha_3)}dt+C_2,$$
$$A_1=\sigma_8(t)\left((\sigma_5(t))^2+(\sigma_6(t))^2\right)\left(\alpha_1{}^4\phi_2+\alpha_3{}^4\phi_2+\alpha_1{}^2\phi_3+\alpha_3{}^2\phi_3+\beta_1{}^2\phi_4+\beta_3{}^2\phi_4+\delta_1{}^2\phi_5+\phi_5\delta_3{}^2\right),$$
$$A_2=-\left(\frac{d}{dt}\sigma_8(t)\right)\left((\sigma_5(t))^2+(\sigma_6(t))^2\right)(\alpha_1-\alpha_3),$$
$$A_3=\sigma_8(t)(\alpha_1-\alpha_3)\left((\sigma_5(t))^2\frac{d}{dt}\sigma_1(t)+\left(\frac{d}{dt}\sigma_1(t)\right)\cdot(\sigma_6(t))^2+\sigma_5(t)\frac{d}{dt}\sigma_5(t)+\sigma_6(t)\frac{d}{dt}\sigma_6(t)\right),$$
$$A_4=-2\sigma_8(t)\left((\sigma_5(t))^2+(\sigma_6(t))^2\right)\left(2\alpha_1{}^3\alpha_3\phi_2-3\alpha_1{}^2\alpha_3{}^2\phi_2+2\alpha_1\alpha_3{}^3\phi_2+\alpha_1\alpha_3\phi_3+\beta_1\beta_3\phi_4+\delta_1\phi_5\delta_3\right), \tag{44}$$

where $\sigma_1(t),\sigma_4(t)$ free functions of t and $\alpha_l,\beta_l,\delta_l,(l=1,2,3,4)$, are free values. It concludes along with the bilinear equation; the exact solution will be as

$$\Sigma_4=\frac{12\phi_2}{\phi_1}(\ln g)_{xx},\ g=e^{\alpha_1x+\beta_1y+\delta_1z+\sigma_1(t)}(\sigma_5(t)\cos(\sigma_2(t))+\sigma_6(t)\sin(\sigma_2(t)))+e^{\alpha_3x+\beta_3y+\delta_3z+\sigma_3(t)}\cdot(\sigma_7(t)\cos(\sigma_4(t))+\sigma_8(t)\sin(\sigma_4(t))). \tag{45}$$

Set II-I:

$$\sigma_2(t)=C_1,$$
$$\sigma_5(t)=0,$$
$$\sigma_8(t)=-\tan(-\sigma_4(t)+C_3)\sigma_7(t),$$
$$\alpha_2=\beta_2=\delta_2=\delta_4=0,$$
$$\sigma_3(t)=\int-\frac{A_1+A_2}{\sigma_6(t)\sigma_7(t)(\alpha_1-\alpha_3)}dt+C_2,$$
$$A_1=-\sigma_6(t)\sigma_7(t)\left(\phi_2(\alpha_1-\alpha_3)^4+\phi_3(\alpha_1-\alpha_3)^2+\phi_4(\beta_1-\beta_3)^2+\phi_5(\delta_1-\delta_3)^2\right),$$
$$A_2=-(\alpha_1-\alpha_3)\left(\left(\frac{d}{dt}\sigma_1(t)\right)\sigma_6(t)\sigma_7(t)-\sigma_8(t)\sigma_6(t)\frac{d}{dt}\sigma_4(t)-\sigma_6(t)\frac{d}{dt}\sigma_7(t)+\left(\frac{d}{dt}\sigma_6(t)\right)\sigma_7(t)\right), \tag{46}$$

where $\sigma_1(t),\sigma_4(t)$ free functions of t and $\alpha_l,\beta_l,\delta_l,(l=1,2,3)$, are free values. It concludes along with the bilinear equation; the exact solution will be as

$$\Sigma_5=\frac{12\phi_2}{\phi_1}(\ln g)_{xx},$$
$$g=e^{\alpha_1x+\beta_1y+\delta_1z+\sigma_1(t)}(\sigma_5(t)\cos(\sigma_2(t))+\sigma_6(t)\sin(\sigma_2(t)))+e^{\alpha_3x+\beta_3y+\delta_3z+\sigma_3(t)}(\sigma_7(t)\cos(\sigma_4(t))+\sigma_8(t)\sin(\sigma_4(t))). \tag{47}$$

Set II-II:

$$\sigma_2(t)=-\arctan\left(\frac{\sigma_5(t)}{\sigma_6(t)}\right)+C_1,$$
$$\sigma_8(t)=-\tan(-\sigma_4(t)+C_3)\sigma_7(t),$$
$$\alpha_2=\beta_2=\delta_2=\delta_4=0,$$
$$\sigma_3(t)=\int-\frac{A_3+A_1+A_2}{\sigma_7(t)\left((\sigma_5(t))^2+(\sigma_6(t))^2\right)(\alpha_1-\alpha_3)}dt+C_2,$$
$$A_1=\left(\frac{d}{dt}\sigma_7(t)\right)\left((\sigma_5(t))^2+(\sigma_6(t))^2\right)(\alpha_1-\alpha_3),$$
$$A_2=-(\alpha_1-\alpha_3)\left(\left((\sigma_5(t))^2+(\sigma_6(t))^2\right)\left(\left(\frac{d}{dt}\sigma_1(t)\right)\sigma_7(t)-\sigma_8(t)\frac{d}{dt}\sigma_4(t)\right)+\sigma_5(t)\left(\frac{d}{dt}\sigma_5(t)\right)\sigma_7(t)+\sigma_6(t)\left(\frac{d}{dt}\sigma_6(t)\right)\sigma_7(t)\right),$$
$$A_3=-\sigma_7(t)\left((\sigma_5(t))^2+(\sigma_6(t))^2\right)\left(\phi_2(\alpha_1-\alpha_3)^4+\phi_3(\alpha_1-\alpha_3)^2+\phi_4(\beta_1-\beta_3)^2+\phi_5(\delta_1-\delta_3)^2\right), \tag{48}$$

where $\sigma_1(t),\sigma_4(t)$ free functions of t and $\alpha_l,\beta_l,\delta_l,(l=1,2,3,4)$, are free values. It concludes along with the bilinear equation; the exact solution will be as

$$\Sigma_6=\frac{12\phi_2}{\phi_1}(\ln g)_{xx},\ g=e^{\alpha_1x+\beta_1y+\delta_1z+\sigma_1(t)}(\sigma_5(t)\cos(\sigma_2(t))+\sigma_6(t)\sin(\sigma_2(t)))+e^{\alpha_3x+\beta_3y+\delta_3z+\sigma_3(t)}(\sigma_7(t)\cos(\sigma_4(t))+\sigma_8(t)\sin(\sigma_4(t))). \tag{49}$$

Figures 7 and 8, respectively, show the analysis of treatment of periodic wave solution with graphs of Σ with the below-selected parameters:

$$C_1=C_2=1,C_3=0.2,\alpha_1=\delta_1=0.2,\beta_1=\alpha_3=0.1,\beta_3=0.3,\delta_3=0.4,$$

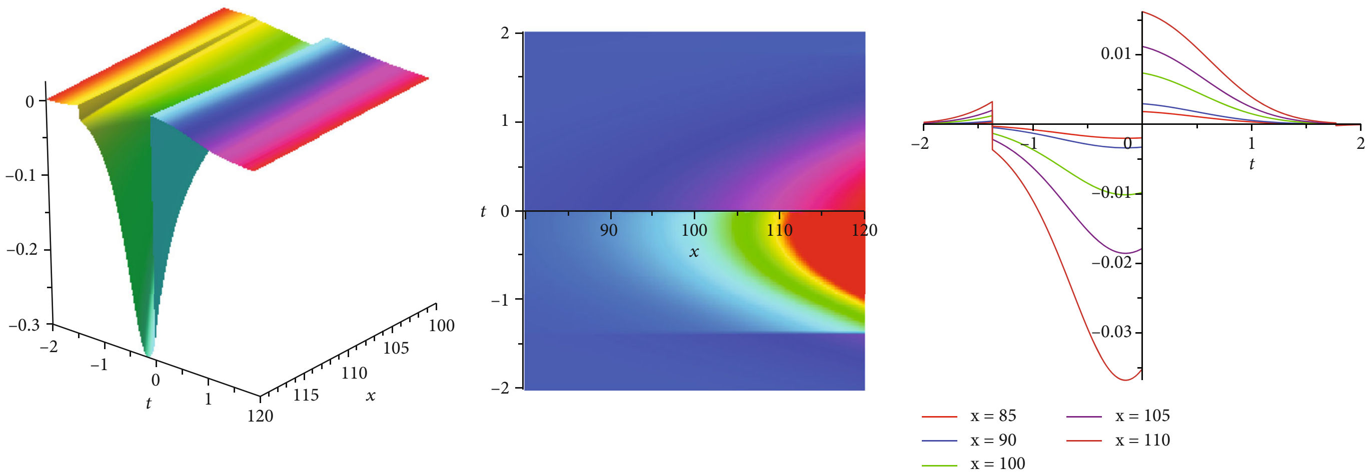

Figure 7: Plot of periodic wave solution (49) (Σ_6) (3D plot, density plot, and 2D plot t), respectively.

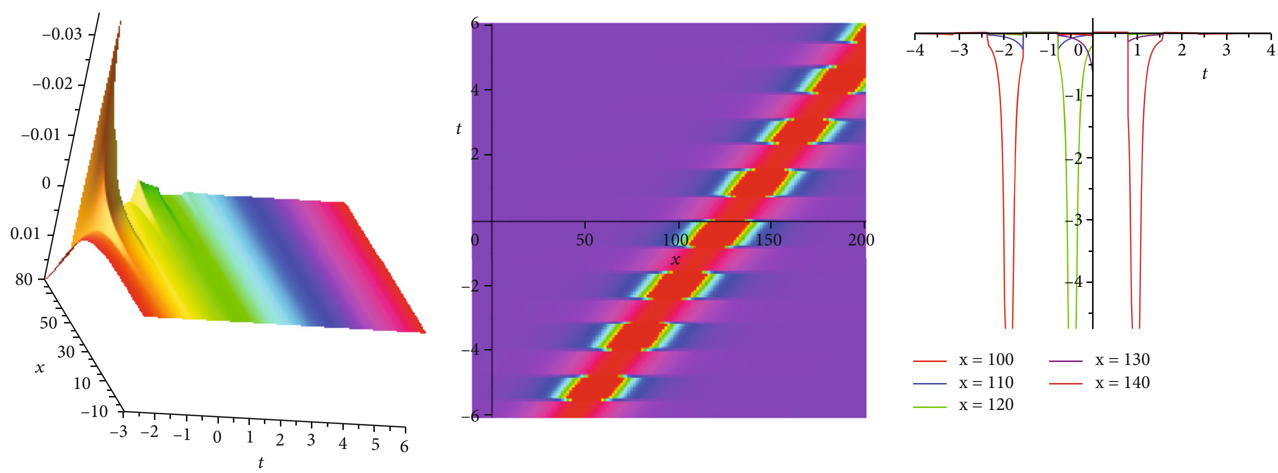

Figure 8: Plot of periodic wave solution (49) (Σ_6) (3D plot, density plot, and 2D plot t), respectively.

$$\begin{aligned}&\sigma_1(t)=\sigma_5(t)=\sigma_6(t)=\sigma_7(t)=\phi_1=\phi_2=\phi_3=t,\ \phi_4=t+1,\\&\quad\phi_5=2t,\ z=1,\ y=1,\\&C_1=C_2=1,\ C_3=0.2,\ \alpha_1=\delta_1=0.2,\ \beta_1=\alpha_3=0.1,\ \beta_3=0.3,\\&\quad\delta_3=0.4,\ \sigma_1(t)=\sin(t),\ \sigma_4(t)=\sin(2t),\\&\sigma_5(t)=\sin(3t),\ \sigma_6(t)=\sin(4t),\ \sigma_7(t)=\sin(t),\ \phi_1=3,\\&\quad\phi_2=2,\ \phi_3=2,\ \phi_4=1,\ \phi_5=2,\ z=1,\ y=1,\end{aligned}\tag{50}$$

in equation (49).

4.2.3. Set III Solutions. III-I:

$$\begin{aligned}\Sigma_7&=\frac{12\phi_2}{\phi_1}(\ln g)_{xx},\ g=e^{\alpha_1x+\beta_1y+\delta_1z+\sigma_1(t)}\sigma_6(t)\sin\left(\beta_2y+\frac{\beta_2\delta_4z}{\beta_4}+2\frac{\beta_2\delta_4\phi_5(\beta_1\delta_4-\beta_3\delta_4-\beta_4\delta_1+\beta_4\delta_3)t}{\beta_4^{\,2}(\alpha_1-\alpha_3)}+C_1\right)\\&\quad+e^{\alpha_3x+\beta_3y+\delta_3z-\int A_1+A_2/\sigma_8(t)\sigma_6(t)\beta_4^{\,2}(\alpha_1-\alpha_3)\,dt}\cdot\sigma_8(t)\sin\left(\beta_4y+\delta_4z+2\frac{\delta_4\phi_5(\beta_1\delta_4-\beta_3\delta_4-\beta_4\delta_1+\beta_4\delta_3)t}{\beta_4(\alpha_1-\alpha_3)}+C_3\right),\\A_1&=-\sigma_6(t)\sigma_8(t)\left(\beta_4^{\,2}\phi_2(\alpha_1-\alpha_3)^4+\beta_4^{\,2}\phi_3(\alpha_1-\alpha_3)^2-\phi_5\left((\beta_1\delta_4-\beta_3\delta_4)^2-(\beta_4\delta_1-\beta_4\delta_3)^2\right)\right),\\A_2&=-\beta_4^{\,2}(\alpha_1-\alpha_3)\left(\sigma_6(t)\sigma_8(t)\frac{d}{dt}\sigma_1(t)-\sigma_6(t)\frac{d}{dt}\sigma_8(t)+\sigma_8(t)\frac{d}{dt}\sigma_6(t)\right).\end{aligned}\tag{51}$$

III-II:

$$\Sigma_8=\frac{12\phi_2}{\phi_1}(\ln g)_{xx},\ g=e^{\alpha_1x+\beta_1y+\delta_1z+\sigma_1(t)}$$

$$
\begin{aligned}
&\cdot\left(\sigma_5(t)\cos\left(\beta_2 y+\frac{\beta_2\delta_4 z}{\beta_4}-\arctan\left(\frac{\sigma_5(t)}{\sigma_6(t)}\right)+A_1\right)\right.\\
&\left.+\sigma_6(t)\sin\left(\beta_2 y+\frac{\beta_2\delta_4 z}{\beta_4}-\arctan\left(\frac{\sigma_5(t)}{\sigma_6(t)}\right)+A_1\right)\right)\\
&+e^{\alpha_3 x+\beta_3 y+\delta_3 z-\int A_2+A_3/\sigma_8(t)\beta_4{}^2\left((\sigma_5(t))^2+(\sigma_6(t))^2\right)(\alpha_1-\alpha_3)\,dt}\\
&\cdot\sigma_8(t)\sin\left(\beta_4 y+\delta_4 z\right.\\
&\left.+2\,\frac{\delta_4\phi_5(\beta_1\delta_4-\beta_3\delta_4-\beta_4\delta_1+\beta_4\delta_3)t}{\beta_4(\alpha_1-\alpha_3)}+C_3\right),\\
A_1&=2\,\frac{\beta_2\delta_4\phi_5(\beta_1\delta_4-\beta_3\delta_4-\beta_4\delta_1+\beta_4\delta_3)t}{\beta_4{}^2(\alpha_1-\alpha_3)},\\
A_2&=-\beta_4{}^2(\alpha_1-\alpha_3)\left(\left((\sigma_5(t))^2+(\sigma_6(t))^2\right)\right.\\
&\cdot\left(\sigma_8(t)\frac{d}{dt}\sigma_1(t)-\frac{d}{dt}\sigma_8(t)\right)+\sigma_6(t)\sigma_8(t)\frac{d}{dt}\sigma_6(t)\\
&\left.+\sigma_8(t)\sigma_5(t)\frac{d}{dt}\sigma_5(t)\right),\\
A_3&=-\sigma_8(t)\left((\sigma_5(t))^2+(\sigma_6(t))^2\right)\left(\beta_4{}^2\phi_2(\alpha_1-\alpha_3)^4\right.\\
&+\beta_4{}^2\phi_3\left((\alpha_1-\alpha_3)^2\right)\left(-\phi_5(\beta_1\delta_4-\beta_3\delta_4)^2\right.\\
&\left.\left.-(\beta_4\delta_1-\beta_4\delta_3)^2\right)\right).
\end{aligned}
\tag{52}
$$

III-III:

$$
\begin{aligned}
\Sigma_9&=\frac{12\phi_2}{\phi_1}(\ln g)_{xx},\ g=e^{\alpha_1 x+\beta_1 y+\delta_1 z+\sigma_1(t)}\sigma_6(t)\sin\\
&\cdot\left(\beta_2 y+\frac{\beta_2\delta_4 z}{\beta_4}\right.\\
&\left.+2\,\frac{\beta_2\delta_4\phi_5(\beta_1\delta_4-\beta_3\delta_4-\beta_4\delta_1+\beta_4\delta_3)t}{\beta_4 A_0}+C_1\right)\\
&+e^{\alpha_3 x+\beta_3 y+\delta_3 z-\int A_2+A_3/\sigma_6(t)\sigma_7(t)\beta_4 A_0\,dt}\\
&\cdot\left(\sigma_7(t)\cos\left(\beta_4 y+\delta_4 z+\sigma_4(t)\right)\right.\\
&\left.-\tan\left(C_3-\int\frac{A_1}{A_0}\,dt\right)\sigma_7(t)\sin\left(\beta_4 y+\delta_4 z+\sigma_4(t)\right)\right),\\
A_0&=\beta_4(\alpha_1-\alpha_3),\\
A_1&=-2\,\delta_4\phi_5(\beta_1\delta_4-\beta_3\delta_4-\beta_4\delta_1+\beta_4\delta_3)+A_0\frac{d}{dt}\sigma_4(t),\\
A_2&=-\beta_4{}^2(\alpha_1-\alpha_3)\left(\sigma_7(t)\sigma_6(t)\frac{d}{dt}\sigma_1(t)\right.\\
&-\left(\frac{d}{dt}\sigma_4(t)\right)\sigma_6(t)\sigma_8(t)+\sigma_7(t)\frac{d}{dt}\sigma_6(t)\\
&\left.-\sigma_6(t)\frac{d}{dt}\sigma_7(t)\right),\\
A_3&=-\sigma_6(t)\left(\sigma_7(t)\beta_4{}^2\phi_2(\alpha_1-\alpha_3)^4+\sigma_7(t)\beta_4{}^2\phi_3(\alpha_1-\alpha_3)^2\right.\\
&-\phi_5(\beta_1\delta_4-\beta_3\delta_4-\beta_4\delta_1+\beta_4\delta_3)(\sigma_7(t)\beta_1\delta_4\\
&\left.-\sigma_7(t)\beta_3\delta_4+\sigma_7(t)\beta_4\delta_1-\sigma_7(t)\beta_4\delta_3-2\,\sigma_8(t)\beta_4\delta_4)\right).
\end{aligned}
\tag{53}
$$

III-IV:

$$
\begin{aligned}
\Sigma_{10}&=\frac{12\phi_2}{\phi_1}(\ln g)_{xx},\\
g&=e^{\alpha_1 x+\beta_1 y+\delta_1 z+\sigma_1(t)}\left(\sigma_5(t)\cos(X_1)+\sigma_6(t)\sin(X_1)\right)\\
&+e^{\alpha_3 x+\beta_3 y+\delta_3 z-\int A_3+A_4/\sigma_7(t)\beta_4{}^2\left((\sigma_5(t))^2+(\sigma_6(t))^2\right)(\alpha_1-\alpha_3)\,dt}\\
&\cdot\left(\sigma_7(t)\cos(X_2)-\tan\left(C_3-\int\frac{A_1}{\beta_4(\alpha_1-\alpha_3)}\,dt\right)\right.\\
&\left.\cdot\sigma_7(t)\sin(X_2)\right),\\
X_1&=\beta_2 y+\frac{\beta_2\delta_4 z}{\beta_4}-\arctan\left(\frac{\sigma_5(t)}{\sigma_6(t)}\right)+A_2 t,\\
X_2&=\beta_4 y+\delta_4 z+\sigma_4(t),\\
A_1&=-2\,\delta_4\phi_5(\beta_1\delta_4-\beta_3\delta_4-\beta_4\delta_1+\beta_4\delta_3)\\
&+\beta_4(\alpha_1-\alpha_3)\frac{d}{dt}\sigma_4(t),\\
A_2&=2\,\frac{\beta_2\delta_4\phi_5(\beta_1\delta_4-\beta_3\delta_4-\beta_4\delta_1+\beta_4\delta_3)}{\beta_4{}^2(\alpha_1-\alpha_3)},\\
A_3&=-\left((\sigma_5(t))^2+(\sigma_6(t))^2\right)\left(\sigma_7(t)\beta_4{}^2\phi_2(\alpha_1-\alpha_3)^4\right.\\
&+\sigma_7(t)\beta_4{}^2\phi_3(\alpha_1-\alpha_3)^2-\phi_5(\beta_1\delta_4-\beta_3\delta_4-\beta_4\delta_1\\
&+\beta_4\delta_3)(\sigma_7(t)\beta_1\delta_4-\sigma_7(t)\beta_3\delta_4+\sigma_7(t)\beta_4\delta_1\\
&\left.-\sigma_7(t)\beta_4\delta_3-2\,\sigma_8(t)\beta_4\delta_4)\right),\\
A_4&=-\beta_4{}^2(\alpha_1-\alpha_3)\left(\left((\sigma_5(t))^2+(\sigma_6(t))^2\right)\right.\\
&\cdot\left(\sigma_7(t)\frac{d}{dt}\sigma_1(t)-\left(\frac{d}{dt}\sigma_4(t)\right)\sigma_8(t)-\frac{d}{dt}\sigma_7(t)\right)\\
&\left.+\left(\frac{d}{dt}\sigma_5(t)\right)\sigma_7(t)\sigma_5(t)+\sigma_7(t)\sigma_6(t)\frac{d}{dt}\sigma_6(t)\right).
\end{aligned}
\tag{54}
$$

Figure 9 shows the analysis of treatment of periodic wave solution with graphs of Σ with the below-selected parameters:

$$
\begin{aligned}
&C_1=C_2=1, C_3=0.2, \alpha_1=\delta_1=0.2, \beta_1=\alpha_3=0.1,\\
&\beta_3=0.3, \delta_3=0.4,
\end{aligned}
\tag{55}
$$

in Equation (54).

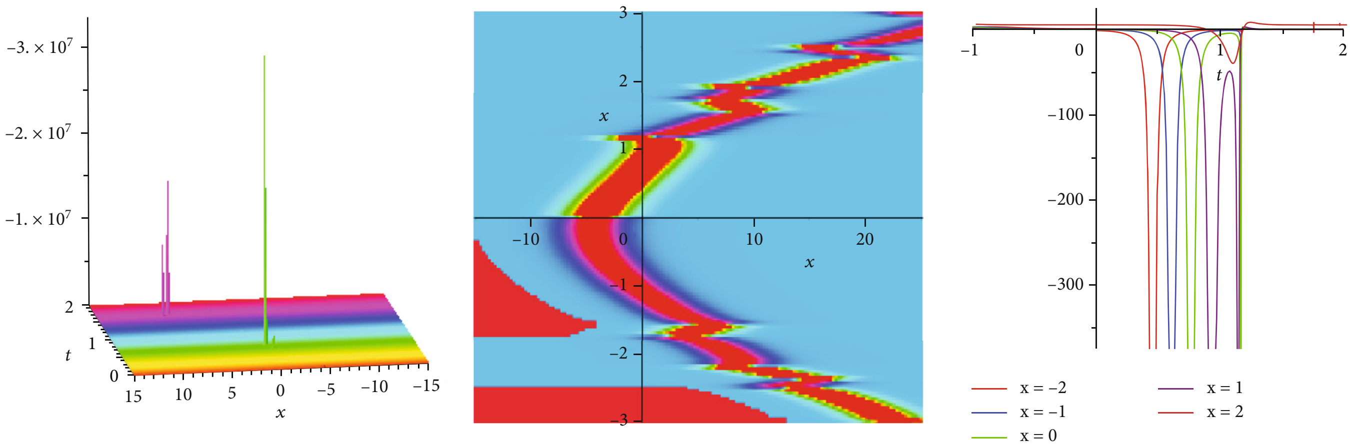

Figure 9: Plot of periodic wave solution (54) (Σ_{10}) (3D plot, density plot, and 2D plot t), respectively.

4.2.4. Set IV Solutions. IV-I:

IV-II:

$$\Sigma_{11} = \frac{12\phi_2}{\phi_1}(\ln g)_{xx},\ g = e^{((\sigma_8(t)\alpha_3-\alpha_4\sigma_7(t))x/\sigma_8(t))+\beta_1 y+\delta_1 z+\sigma_1(t)}\sigma_6(t)\sin(C_1) + e^{\alpha_3 x+\beta_3 y+\delta_3 z+2(t(\beta_1\phi_4\beta_4-\beta_3\phi_4\beta_4+\delta_1\delta_4\phi_5-\delta_3\delta_4\phi_5)/\alpha_4)+\sigma_1(t)+\ln(\sigma_6(t)/\sigma_8(t))}\sigma_8(t) \times \sin\left(\alpha_4 x+\beta_4 y+\delta_4 z - \frac{\left(3\phi_3\alpha_4^{2}-\phi_4\left(4\beta_1^{2}-8\beta_1\beta_3+4\beta_3^{2}-3\beta_4^{2}\right)-\phi_5\left(4\delta_1^{2}-8\delta_1\delta_3+4\delta_3^{2}-3\delta_4^{2}\right)\right)t}{3\alpha_4}\right), \quad (56)$$

$$\phi_2 = \frac{1}{3}\frac{\phi_4(\beta_1-\beta_3)^2+\phi_5(\delta_1-\delta_3)^2}{\alpha_4^{4}}.$$

IV-III:

$$\Sigma_{12} = \frac{12\phi_2}{\phi_1}(\ln g)_{xx},$$

$$g = e^{((\sigma_8(t)\alpha_3-\alpha_4\sigma_7(t))x/\sigma_8(t))+\beta_1 y+\delta_1 z+\sigma_1(t)}\left(\sigma_5(t)\cos\left(C_1-\arctan\left(\frac{\sigma_5(t)}{\sigma_6(t)}\right)\right)+\sigma_6(t)\sin\left(C_1-\arctan\left(\frac{\sigma_5(t)}{\sigma_6(t)}\right)\right)\right) + e^{\alpha_3 x+\beta_3 y+\delta_3 z+\int\left((A_2+A_1)/\left(\sigma_8(t)\alpha_4\left((\sigma_5(t))^2+(\sigma_6(t))^2\right)\right)\right)dt}\sigma_8(t) \times \sin\left(\alpha_4 x+\beta_4 y+\delta_4 z - \frac{\left(3\phi_3\alpha_4^{2}-\phi_4\left(4\beta_1^{2}-8\beta_1\beta_3+4\beta_3^{2}-3\beta_4^{2}\right)-\phi_5\left(4\delta_1^{2}-8\delta_1\delta_3+4\delta_3^{2}-3\delta_4^{2}\right)\right)t}{3\alpha_4}\right), \quad (57)$$

$$A_1 = \alpha_4\left(\left((\sigma_5(t))^2+(\sigma_6(t))^2\right)\left(\sigma_8(t)\frac{d}{dt}\sigma_1(t)-\frac{d}{dt}\sigma_8(t)\right)+\sigma_8(t)\left(\sigma_5(t)\frac{d}{dt}\sigma_5(t)+\sigma_6(t)\frac{d}{dt}\sigma_6(t)\right)\right),$$

$$A_2 = 2\sigma_8(t)\left((\sigma_5(t))^2+(\sigma_6(t))^2\right)(\beta_1\phi_4\beta_4-\beta_3\phi_4\beta_4+\delta_1\delta_4\phi_5-\delta_3\delta_4\phi_5),\ \phi_2 = \frac{\phi_4(\beta_1-\beta_3)^2+\phi_5(\delta_1-\delta_3)^2}{3\alpha_4^{4}}.$$

IV-IV:

$$
\begin{aligned}
\Sigma_{13} &= \frac{12\phi_2}{\phi_1}(\ln g)_{xx}, \\
g &= e^{((\sigma_8(t)\alpha_3-\alpha_4\sigma_7(t))x/\sigma_8(t))+\beta_1 y+\delta_1 z+\sigma_1(t)}(\sigma_5(t)\cos(\sigma_2(t))+\sigma_6(t)\sin(\sigma_2(t))) \\
&\quad + e^{\alpha_3 x+\beta_3 y+\delta_3 z+\sigma_3(t)}(\sigma_7(t)\cos(\alpha_4 x+\beta_4 y+\delta_4 z+\sigma_4(t))+\sigma_8(t)\sin(\alpha_4 x+\beta_4 y+\delta_4 z+\sigma_4(t))), \\
\sigma_2(t) &= C_1, \\
\sigma_3(t) &= \int -\frac{1}{3}\,\frac{A_2+A_3}{\sigma_7(t)\sigma_8(t)\sigma_6(t)\alpha_4\left((\sigma_7(t))^2+(\sigma_8(t))^2\right)}\,dt + C_2, \\
\sigma_8(t) &= \int \frac{A_1 - 3\,\alpha_4\left((\sigma_7(t))^2+(\sigma_8(t))^2\right)\left((\sigma_8(t))^2(d/dt)\sigma_4(t)+(\sigma_7(t))^2(d/dt)\sigma_4(t)+\sigma_8(t)(d/dt)\sigma_7(t)\right)}{-3\,(\sigma_7(t))^3\alpha_4 - 3\,\sigma_7(t)(\sigma_8(t))^2\alpha_4}\,dt, \\
A_1 &= -(\sigma_8(t))^4\left(3\,\phi_3\alpha_4{}^2 - 4\,\beta_1{}^2\phi_4 + 8\,\beta_1\phi_4\beta_3 - 4\,\phi_4\beta_3{}^2 + 3\,\beta_4{}^2\phi_4 - 4\,\delta_1{}^2\phi_5 + 8\,\delta_1\phi_5\delta_3 - 4\,\phi_5\delta_3{}^2 + 3\,\delta_4{}^2\phi_5\right) \\
&\quad + 2\,(\sigma_8(t))^2\sigma_7(t)\left(4\,\sigma_8(t)\beta_1\beta_4\phi_4 - 4\,\sigma_8(t)\beta_3\beta_4\phi_4 + 4\,\sigma_8(t)\delta_1\delta_4\phi_5 - 4\,\sigma_8(t)\delta_3\delta_4\phi_5 - 3\,\sigma_7(t)\alpha_4{}^2\phi_3 - \sigma_7(t)\beta_4{}^2\phi_4\right. \\
&\quad \left. - \sigma_7(t)\delta_4{}^2\phi_5\right) - 3\,(\sigma_7(t))^4\left(\phi_3\alpha_4{}^2 + \beta_4{}^2\phi_4 + \delta_4{}^2\phi_5\right), \\
A_2 &= 3\,\sigma_8(t)\alpha_4\left((\sigma_7(t))^2+(\sigma_8(t))^2\right)\left(\sigma_8(t)\sigma_6(t)\frac{d}{dt}\sigma_4(t) - \sigma_7(t)\left(\frac{d}{dt}\sigma_1(t)\right)\sigma_6(t) - \sigma_7(t)\frac{d}{dt}\sigma_6(t) + \sigma_6(t)\frac{d}{dt}\sigma_7(t)\right), \\
A_3 &= \sigma_6(t)\left(3\,\alpha_4{}^2\phi_3\left((\sigma_7(t))^2+(\sigma_8(t))^2\right)^2 - \phi_4\left((\sigma_8(t))^4\left(4\,\beta_1{}^2 - 8\,\beta_1\beta_3 + 4\,\beta_3{}^2 - 3\,\beta_4{}^2\right) + 14\,(\sigma_8(t))^3\sigma_7(t)\beta_4(\beta_1-\beta_3)\right.\right. \\
&\quad - 4\,(\sigma_8(t))^2(\sigma_7(t))^2(-\beta_3+\beta_1-\beta_4)(-\beta_3+\beta_1+\beta_4) - 2\,\sigma_8(t)(\sigma_7(t))^3\beta_4(\beta_1-\beta_3) - \left.\beta_4{}^2(\sigma_7(t))^4\right) \\
&\quad - \phi_5\left((\sigma_8(t))^4\left(4\,\delta_1{}^2 - 8\,\delta_1\delta_3 + 4\,\delta_3{}^2 - 3\,\delta_4{}^2\right) + 14\,(\sigma_8(t))^3\sigma_7(t)\delta_4(\delta_1-\delta_3)\right. \\
&\quad \left.\left. -4\,(\sigma_8(t))^2(\sigma_7(t))^2(-\delta_3+\delta_1-\delta_4)(-\delta_3+\delta_1+\delta_4) - 2\,\sigma_8(t)(\sigma_7(t))^3\delta_4(\delta_1-\delta_3) - \delta_4{}^2(\sigma_7(t))^4\right)\right), \\
\phi_2 &= \frac{1}{3}\,\frac{(\sigma_8(t))^2\left(\phi_4(\beta_1\sigma_8(t)-\beta_3\sigma_8(t)+\beta_4\sigma_7(t))^2 + \phi_5(\delta_1\sigma_8(t)-\delta_3\sigma_8(t)+\delta_4\sigma_7(t))^2\right)}{\alpha_4{}^4\left((\sigma_7(t))^4 + 2\,(\sigma_7(t))^2(\sigma_8(t))^2 + (\sigma_8(t))^4\right)}.
\end{aligned}
\tag{58}
$$

4.2.5. Set V Solutions

$$
\begin{aligned}
\Sigma_{14} &= \frac{12\phi_2}{\phi_1}(\ln g)_{xx},\\
g &= e^{((\sigma_8(t)\alpha_3-\alpha_4\sigma_7(t))x/\sigma_8(t))+\beta_1 y+\delta_1 z+\sigma_1(t)}(\sigma_5(t)\cos(\sigma_2(t))+\sigma_6(t)\sin(\sigma_2(t)))\\
&\quad + e^{\alpha_3 x+\beta_3 y+\delta_3 z+\sigma_3(t)}(\sigma_7(t)\cos(\alpha_4 x+\beta_4 y+\delta_4 z+\sigma_4(t))+\sigma_8(t)\sin(\alpha_4 x+\beta_4 y+\delta_4 z+\sigma_4(t))),\\
\sigma_2(t) &= -\arctan\left(\frac{\sigma_5(t)}{\sigma_6(t)}\right)+C_1,\\
\sigma_8(t) &= \int\frac{A_2+A_1}{-3(\sigma_7(t))^3\alpha_4-3\sigma_7(t)(\sigma_8(t))^2\alpha_4}\,dt,\\
\sigma_3(t) &= \int -\frac{1}{3}\,\frac{A_3+A_4}{\sigma_7(t)\sigma_8(t)\alpha_4\left((\sigma_5(t))^2+(\sigma_6(t))^2\right)\left((\sigma_7(t))^2+(\sigma_8(t))^2\right)}\,dt+C_2,\\
A_1 &= -3\alpha_4\left((\sigma_7(t))^2+(\sigma_8(t))^2\right)\left((\sigma_8(t))^2\frac{d}{dt}\sigma_4(t)+(\sigma_7(t))^2\frac{d}{dt}\sigma_4(t)+\sigma_8(t)\frac{d}{dt}\sigma_7(t)\right),\\
A_2 &= -3\alpha_4{}^2\phi_3\left((\sigma_7(t))^2+(\sigma_8(t))^2\right)^2+\phi_4\left((\sigma_8(t))^4\left(4\beta_1{}^2-8\beta_1\beta_3+4\beta_3{}^2-3\beta_4{}^2\right)+\sigma_7(t)\beta_4\left(8(\sigma_8(t))^3\beta_1-8(\sigma_8(t))^3\beta_3\right.\right.\\
&\quad \left.\left.-2(\sigma_8(t))^2\sigma_7(t)\beta_4-3(\sigma_7(t))^3\beta_4\right)\right)+\phi_5\left((\sigma_8(t))^4\left(4\delta_1{}^2-8\delta_1\delta_3+4\delta_3{}^2-3\delta_4{}^2\right)+\sigma_7(t)\delta_4\left(8(\sigma_8(t))^3\delta_1\right.\right.\\
&\quad \left.\left.-8(\sigma_8(t))^3\delta_3-2(\sigma_8(t))^2\sigma_7(t)\delta_4-3(\sigma_7(t))^3\delta_4\right)\right),\\
A_3 &= 3\sigma_8(t)\alpha_4\left((\sigma_7(t))^2+(\sigma_8(t))^2\right)\left(\left((\sigma_5(t))^2+(\sigma_6(t))^2\right)\left(\sigma_8(t)\frac{d}{dt}\sigma_4(t)-\sigma_7(t)\frac{d}{dt}\sigma_1(t)+\frac{d}{dt}\sigma_7(t)\right)\right.\\
&\quad \left.-\sigma_7(t)\left(\sigma_5(t)\frac{d}{dt}\sigma_5(t)+\sigma_6(t)\frac{d}{dt}\sigma_6(t)\right)\right),\\
A_4 &= \left((\sigma_5(t))^2+(\sigma_6(t))^2\right)\left((\sigma_8(t))^4\left(3\phi_3\alpha_4{}^2-4\beta_1{}^2\phi_4+8\beta_1\phi_4\beta_3-4\phi_4\beta_3{}^2+3\beta_4{}^2\phi_4-4\delta_1{}^2\phi_5+8\delta_1\phi_5\delta_3\right.\right.\\
&\quad \left.-4\phi_5\delta_3{}^2+3\delta_4{}^2\phi_5\right)+(\sigma_7(t))^4\left(3\phi_3\alpha_4{}^2+\beta_4{}^2\phi_4+\delta_4{}^2\phi_5\right)+2(\sigma_8(t))^2(\sigma_7(t))^2\left(3\phi_3\alpha_4{}^2+2\beta_1{}^2\phi_4+2\phi_4\beta_3{}^2\right.\\
&\quad \left.-2\beta_4{}^2\phi_4+2\delta_1{}^2\phi_5-2\delta_4{}^2\phi_5\right)+2(\sigma_8(t))^2(\sigma_7(t))^2\left(3\phi_3\alpha_4{}^2+2\beta_1{}^2\phi_4+2\phi_4\beta_3{}^2-2\beta_4{}^2\phi_4+2\delta_1{}^2\phi_5\right.\\
&\quad \left.-4\delta_1\phi_5\delta_3+2\phi_5\delta_3{}^2-2\delta_4{}^2\phi_5\right)-14(\sigma_8(t))^3\sigma_7(t)(\beta_1\phi_4\beta_4-\beta_3\phi_4\beta_4+\delta_1\delta_4\phi_5\\
&\quad \left.-\delta_3\delta_4\phi_5)-2\sigma_8(t)(\sigma_7(t))^2(4\sigma_8(t)\beta_1\beta_3\phi_4-\sigma_7(t)\beta_1\beta_4\phi_4+\sigma_7(t)\beta_3\beta_4\phi_4-\sigma_7(t)\delta_1\delta_4\phi_5)\right),\\
\phi_2 &= \frac{1}{3}\,\frac{(\sigma_8(t))^2\left(\phi_4(\beta_1\sigma_8(t)-\beta_3\sigma_8(t)+\beta_4\sigma_7(t))^2+\phi_5(\delta_1\sigma_8(t)-\delta_3\sigma_8(t)+\delta_4\sigma_7(t))^2\right)}{\alpha_4{}^4\left((\sigma_7(t))^4+2(\sigma_7(t))^2(\sigma_8(t))^2+(\sigma_8(t))^4\right)}.
\end{aligned}
\tag{59}
$$

$$
\begin{aligned}
\Sigma_{15} &= \frac{12\phi_2}{\phi_1}(\ln g)_{xx},\\
g &= e^{X_1+\sigma_1(t)}\left(\sigma_5(t)\cos\left(\alpha_2 x + \frac{\alpha_2\beta_4 y}{\alpha_4} + \delta_2 z + \sigma_2(t)\right) + \sigma_6(t)\sin\left(\alpha_2 x + \frac{\alpha_2\beta_4 y}{\alpha_4} + \delta_2 z + \sigma_2(t)\right)\right)\\
&\quad + e^{\alpha_3 x+\beta_3 y+\delta_3 z+\sigma_3(t)}(\sigma_7(t)\cos(\alpha_4 x + \beta_4 y + \delta_4 z + \sigma_4(t)) + \sigma_8(t)\sin(\alpha_4 x + \beta_4 y + \delta_4 z + \sigma_4(t))),\\
X_1 &= \frac{(\sigma_8(t)(\alpha_2\sigma_5(t) + \alpha_3\sigma_6(t)) - \alpha_4\sigma_6(t)\sigma_7(t))x}{\sigma_6(t)\sigma_8(t)} + \frac{(\sigma_8(t)(\alpha_2\beta_4\sigma_5(t) + \alpha_4\beta_3\sigma_6(t)) - \alpha_4\beta_4\sigma_6(t)\sigma_7(t))y}{\alpha_4\sigma_6(t)\sigma_8(t)}\\
&\quad + \frac{1}{2}\frac{(B_1+B_2+B_3)z}{\alpha_2\alpha_4\sigma_6(t)\sigma_8(t)(\alpha_2\sigma_5(t)\sigma_8(t) - \alpha_4\sigma_6(t)\sigma_7(t))},\\
B_1 &= \left((\alpha_2\delta_4 + \alpha_4\delta_2)(\sigma_5(t))^2 + 2\sigma_5(t)\sigma_6(t)\alpha_4\delta_3 + (\sigma_6(t))^2(\alpha_2\delta_4 - \alpha_4\delta_2)\right)(\sigma_8(t))^2{\alpha_2}^2,\\
B_2 &= -2\sigma_7(t)\sigma_5(t)\sigma_8(t)\sigma_6(t)\alpha_2\alpha_4(\alpha_2\delta_4 + \alpha_4\delta_2),\\
B_3 &= (\sigma_6(t))^2{\alpha_4}^2\left((\sigma_7(t))^2(\alpha_2\delta_4 + \alpha_4\delta_2) - 2\sigma_7(t)\sigma_8(t)\alpha_2\delta_3 - (\sigma_8(t))^2(\alpha_2\delta_4 - \alpha_4\delta_2)\right),\\
\sigma_2(t) &= \int -\frac{A_1 + A_2}{(\sigma_6(t))^2(\sigma_8(t))^2{\alpha_4}^2(\alpha_2\delta_4 - \alpha_4\delta_2)^2\left((\sigma_6(t))^2 + (\sigma_5(t))^2\right)}\,dt + C_1,\\
A_1 &= (\sigma_8(t))^2(\sigma_6(t))^2{\alpha_4}^2(\alpha_2\delta_4 - \alpha_4\delta_2)^2\left(\left(\frac{d}{dt}\sigma_5(t)\right)\sigma_6(t) - \left(\frac{d}{dt}\sigma_6(t)\right)\sigma_5(t)\right),\\
A_2 &= \alpha_2\left((\sigma_6(t))^2 + (\sigma_5(t))^2\right)\left(12\sigma_7(t)\sigma_6(t){\alpha_4}^5{\delta_2}^2\phi_2(\sigma_7(t)\sigma_6(t)\alpha_4 - 2\sigma_5(t)\sigma_8(t)\alpha_2) + (\sigma_8(t))^2\left(4{\alpha_2}^2{\alpha_4}^2\phi_2\left(3(\sigma_5(t))^2{\alpha_4}^2{\delta_2}^2\right.\right.\right.\\
&\quad \left.\left.\left. - (\sigma_6(t))^2{\alpha_2}^2{\delta_4}^2 + 2(\sigma_6(t))^2\alpha_2\alpha_4\delta_2\delta_4 - (\sigma_6(t))^2{\alpha_4}^2{\delta_2}^2\right) + (\sigma_6(t))^2(\alpha_2\delta_4 - \alpha_4\delta_2)^2\left({\alpha_4}^2\phi_3 + {\beta_4}^2\phi_4\right)\right),\\
A_3 &= -(\sigma_8(t))^2(\sigma_6(t))^2\alpha_4(\alpha_2\delta_4 - \alpha_4\delta_2)^2\left(\sigma_7(t)\frac{d}{dt}\sigma_8(t) - \sigma_8(t)\frac{d}{dt}\sigma_7(t)\right),\\
A_4 &= \left((\sigma_8(t))^2 + (\sigma_7(t))^2\right)\left(12{\alpha_2}^2{\alpha_4}^2{\delta_4}^2\phi_2(\sigma_7(t)\sigma_6(t)\alpha_4 - \sigma_5(t)\sigma_8(t)\alpha_2)^2 - (\sigma_8(t))^2(\sigma_6(t))^2(\alpha_2\delta_4 - \alpha_4\delta_2)^2\left(4{\alpha_4}^4\phi_2 - {\alpha_4}^2\phi_3 - {\beta_4}^2\phi_4\right)\right),\\
\sigma_4(t) &= \int -\frac{A_3 + A_4}{(\sigma_6(t))^2(\sigma_8(t))^2\alpha_4(\alpha_2\delta_4 - \alpha_4\delta_2)^2\left((\sigma_8(t))^2 + (\sigma_7(t))^2\right)}\,dt,\\
\sigma_3(t) &= \int -\frac{A_5 + A_6}{(\sigma_6(t))^3(\sigma_8(t))^3{\alpha_4}^2(\alpha_2\delta_4 - \alpha_4\delta_2)^2\left((\sigma_8(t))^2 + (\sigma_7(t))^2\right)\left((\sigma_6(t))^2 + (\sigma_5(t))^2\right)}\,dt,\\
A_5 &= S_1S_2(\sigma_7(t)\sigma_6(t)\alpha_4 - \sigma_5(t)\sigma_8(t)\alpha_2)\left(4\sigma_7(t)\sigma_6(t){\alpha_4}^3\phi_2S_3S_4 + 4(\sigma_5(t))^2(\sigma_8(t))^2{\alpha_2}^2{\alpha_4}^2\phi_2S_3\right.\\
&\quad \left. + (\sigma_8(t))^2(\sigma_6(t))^2(\alpha_2\delta_4 - \alpha_4\delta_2)\left(12\alpha_2{\alpha_4}^3\phi_2(\alpha_2\delta_2 - \alpha_4\delta_4) + \left({\alpha_4}^2\phi_3 + {\beta_4}^2\phi_4\right)(\alpha_2\delta_4 - \alpha_4\delta_2)\right)\right),\\
A_6 &= -(\sigma_8(t))^3(\sigma_6(t))^3{\alpha_4}^2(\alpha_2\delta_4 - \alpha_4\delta_2)^2\left(\left(\frac{d}{dt}\sigma_1(t)\right)S_1S_2 + S_2\left(\left(\frac{d}{dt}\sigma_5(t)\right)\sigma_5(t) + \left(\frac{d}{dt}\sigma_6(t)\right)\sigma_6(t)\right) - S_1\left(\sigma_7(t)\frac{d}{dt}\sigma_7(t) + \sigma_8(t)\frac{d}{dt}\sigma_8(t)\right)\right),\\
S_1 &= (\sigma_6(t))^2 + (\sigma_5(t))^2,\\
S_2 &= (\sigma_8(t))^2 + (\sigma_7(t))^2,\\
S_3 &= {\alpha_2}^2{\delta_4}^2 + \alpha_2\alpha_4\delta_2\delta_4 + {\alpha_4}^2{\delta_2}^2,\\
S_4 &= \sigma_7(t)\sigma_6(t)\alpha_4 - 2\sigma_5(t)\sigma_8(t)\alpha_2,\\
\alpha_1 &= \frac{\sigma_5(t)\sigma_8(t)\alpha_2 + \alpha_3\sigma_6(t)\sigma_8(t) - \sigma_7(t)\sigma_6(t)\alpha_4}{\sigma_6(t)\sigma_8(t)},\\
\phi_5 &= 12\frac{(\sigma_5(t)\sigma_8(t)\alpha_2 - \sigma_7(t)\sigma_6(t)\alpha_4)^2{\alpha_4}^2{\alpha_2}^2\phi_2}{(\alpha_2\delta_4 - \alpha_4\delta_2)^2(\sigma_6(t))^2(\sigma_8(t))^2},\\
\delta_1 &= 1/2\,\frac{(\sigma_8(t))^2{\alpha_2}^2 + (\sigma_6(t))^2{\alpha_4}^2\left((\sigma_7(t))^2\alpha_2\delta_4 + (\sigma_7(t))^2\alpha_4\delta_2 - (\sigma_8(t))^2\alpha_2\delta_4 + (\sigma_8(t))^2\alpha_4\delta_2\right)}{\alpha_2\alpha_4\sigma_6(t)\sigma_8(t)(\sigma_5(t)\sigma_8(t)\alpha_2 - \sigma_7(t)\sigma_6(t)\alpha_4)},\\
H_1 &= (\sigma_5(t))^2\alpha_2\delta_4 + (\sigma_5(t))^2\alpha_4\delta_2 + (\sigma_6(t))^2\alpha_2\delta_4 - (\sigma_6(t))^2\alpha_4\delta_2.
\end{aligned}
\tag{60}
$$

4.2.6. Set VI Solutions

$$\begin{aligned}\Sigma_{16} &= \frac{12\phi_2}{\phi_1}(\ln g)_{xx},\\ g &= e^{\alpha_3 x+\beta_1 y+\delta_1 z+\sigma_1(t)}(\sigma_5(t)\cos(\sigma_2(t))+\sigma_6(t)\sin(\sigma_2(t)))\\ &\quad + e^{\alpha_3 x+\beta_3 y+\delta_3 z+\sigma_3(t)}(\sigma_7(t)\cos(\sigma_4(t))+\sigma_8(t)\sin(\sigma_4(t))).\end{aligned} \tag{61}$$

5. Conclusion

This article investigated the combined damped sinusoidal oscillation solutions to the $(3+1)$-D variable-coefficient generalized nonlinear wave equation. The bilinear form of the equation has been described by means of Hirota derivatives. The governing equation is translated to nonlinear ODE using the Cole-Hopf algorithm. Two types of rational periodic solutions have been constructed, which are bright, singular, and periodic, damped sinusoidal oscillation solitons in terms of trigonometric and rational functions. The dynamic features of different types of traveling waves are analyzed in detail through numerical simulation. Meanwhile, the profiles of the surface for the deduced solutions have been depicted in 2D, density, and 3D for free parameters. From the acquired results, it can be concluded that the procedures followed in this analysis can be implemented in a simple and straightforward manner to create new exact solutions of many other nonlinear partial differential equations in terms of the Hirota operator. It is also a very easy-to-use mathematical tool to solve real-life problems in different areas of engineering and sciences.

Data Availability

The datasets supporting the conclusions of this article are included in the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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