*Research Article*

# Analysis of Reciprocal Thermal Conductivity on Free Convection Flow along a Wavy Vertical Surface

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The effects of thermal conductivity which depend on temperature are conversely proportional with the linear function of temperature on free convective flow where the fluid is viscous and incompressible along a heated uniform and the vertical wavy surface has been examined in this study. The boundary layer equations with the associated boundary conditions that govern the flow are converted into a nondimensional form by using an appropriate transformation. In the domain of a vertical plate that is flat, the resulting method of nonlinear PDEs is mapped and then worked out numerically by applying the implicit central finite difference technique with Newton's quasilinearization method, and the block Thomas algorithm is well known as the Keller-box method. The outputs are obtained in the terms of the heat transferring rate, the frictional coefficient of skin, the isotherms, and streamlines. The outcomes showed that the local heat transferring rate, the local skin friction coefficient, the temperature, and the velocity all are decreasing, and both the thermal layer of boundary and velocity become narrower with the rising values of reciprocal variation of temperature-dependent thermal conductivity. On the other hand, the friction coefficient of skin, the velocity, and the temperature decrease where the friction coefficient of skin and velocity decrease by 43% and 64%, respectively, but the heat transfer rate increases by 61% approximately, and both the boundary layer thermal and velocity become thinner when the Prandtl number increases.

## 1. Introduction

In various segments of engineering and science, boundary layer natural convective flow comes from a surface that is wavy and vertical, which is greatly advantageous. The flow is held by the rough surfaces and changes the heat transfer rate. A kind of heat transfer is natural convection, which happens only for differences in density in fluid because of a temperature gradient. The process of natural convection is governed substantially by three features, which are the temperature difference in the flow field, the body force, and the density of fluid alternatives with temperature. In many applications, the study of heat transferring from a crooked surface is needed too, as these kinds of surfaces are frequently stayed. From irregular surfaces, laminar natural convective flow can be used for transmitting heat in various heat transferring devices, such as flat-plate condensers in refrigerators, flat-plate solar collectors, industrial applica-

tions, and other functional clothing designs. Thermal conductivity is a corporal property which may alternate appreciably with temperature. It is found for liquid that $\kappa$; thermal conductivity differs with temperature where the range is 0-400˚F [1] in an approximate linear manner. Thermal conductivity is one of the key factors in the prediction of flow behavior. Laminar free convection about the Newtonian fluid and transferring heat problems worked by several checkers due to its considering practicable applications. Alam et al. [2] explored free convection when magnetic field was present which was transverse from a vertical surface and wavy. Free convective flow with MHD field in porous channels which was opening ended and vertical was presented by Al-Nimr and Hader [3]. Effects of magnetic field through vertical stratum on over back convection had observed by Ahmed and Zaidi [4].

Alim et al. [5] studied the Joule heating effects of MHD natural convective flow on the combination of conduction

from a flat vertical plate. MHD flow with natural convection, heat generation, and viscous dissipation over a sphere was conducted by Alam et al. [6]. Ahmed et al. [7] investigated numerically the MHD radiative heat and mass transfer of nanofluid flow with viscous dissipation and Joule heating effects towards a vertical wavy surface using the Keller-box method. Bhavnani and Bergles [8] researched free convective temperature transport from sinusoidal surface that was wavy. The impact of gradient physics properties on convective force was analyzed by Charrudeau [9]. Chen and Wang [10] examined the micropolar fluids, the force convection along a surface that was wavy. Free convective MHD flow with a porous infinite plate and visco-elastic fluid was explored by Chowdhury and Islam [11]. Cheng [12] demonstrated the effects of temperature-dependent viscosity from an isothermal horizontal cylinder with an elliptic cross section on natural convective heat transfer. Cengel [13] observed both mass and heat transfer. The flow in a channel under the effects of transverse magnetic field during fluid was studied by Damesh et al. [14]. El-Amin [15] conducted the coupling reaction of viscous dissolution and heating of Joule on MHD force convective over a horizontal cylinder which was nonisothermal that placed in fluid saturated porous medium. Gray et al. [16] investigated the nature convective flows with vertical, deriving from the coupling of buoyancy effects of thermal and mass diffusion. Hady et al. [17] analyzed boundary layer flow of mixed convection with variety of viscosity on a continuous flat plate.

Hadjadj and Kyal [18] investigated the effects of two sinusoidal protuberances in a vertical annulus on free convection. Hossain et al. [19] explored for the fluid, the natural convective flow having thermal conductivity and viscosity which were dependent on temperature, past a permeable wedge. Hossain et al. [20] investigated the natural convection of fluid from a heated vertical wavy surface using temperature-dependent viscosity. Jang and Yan [21, 22] observed mixed and natural convective mass and the transfer of heat across a surface that was vertical but wavy. In a circular cylinder, the magnetohydrodynamics of natural convective heat flow with thermal radiation and heat generation effects saturated by nanofluid was evaluated by Javed et al. [23]. Kays [1] worked with convective mass and heat transfer. The boundary-layer free-convective flow through a wavy vertical surface using non-Newtonian fluid was investigated by Kumari et al. [24]. Mahmud et al. [25] conducted the natural convection with wavy vertical walls within an enclosure. Molla et al. [26] studied natural convective flow together with uniform surface temperature across a vertical surface, which was wavy in the existence of warmth absorption/creation. Nasrin and Alim [27] worked with free convective flow under the influence of an MHD field along with a flat plate that was vertical with viscosity and the thermal conductivity varied with temperature. The variation of viscosity and thermal conductivity in a vertical plate was examined numerically by Palani and Kim [28]. Parveen and Alim [29] explored Joule heat effects on free convective MHD flow of the fluid within viscosity that was temperature dependent, which was inversely proportional to temperature through a vertically wavy surface where temperature was a linear function. Results of viscosity and thermal conductivity dependent on temperature on natural convective MHD flow across a wavy vertical surface were presented by Parveen and Alim [30].

Rahman and Alim [31] observed numerically the magnetohydrodynamic natural convective heat transport flow with temperature-dependent thermal conductivity through a plate that was vertical. Free convection, which was influenced by a vertical surface that was wavy, together with heat flux and uniform over a porous medium, was investigated by Rees and Pop [32]. An asymmetrical wavy motion of blood with convective boundary conditions under the influence of entropy generation was analyzed mathematically by Riaz et al. [33]. Sparrow and Cess [34] investigated the effects of a magnetic field on heat transfer that was free convective. Tashtoush and Al-Odat [35] investigated the effect of a wavy surface magnetic field on fluid flow and heat within a changeable heat flux. Tajul and Parveen [36] investigated natural convective flow across a wavy surface that was vertical in Joule heat presence within the viscous dissipation and magnetic field effects. Tinni and Parveen [37] examined the reciprocal variety of thermal conductivity and viscosity with temperature on free convective flow through a vertical surface that was wavy. In a wavy wall channel, forced convection was explored by Wang and Chen [38]. Wilks [39] presented a strong cross field, magnetohydrodynamic free convection regarding a plate which was vertical and semi-infinite. The natural convective transfer of heat from a wavy vertical surface, which was isothermal, was first observed by Yao [40–42] and used expanded Prandtl's transposition method and finite-difference technique. For studying the natural convective heat transfer, he proposed a simple transformation from an isothermal wavy vertical surface. From beyond discussion, it is clear that for natural convection, thermal conductivity is a physical phenomenon that is macroscopic and very interesting in fluid dynamics. No earlier studies had assumed the reciprocal variety of thermal conductivity on the flow of free convection through a vertical wavy surface.

This learning is to investigate the concept of thermal conductivity effects, which are conversely proportional to a linear function of temperature with natural convection across a wavy vertical surface. It is still established that thermal conductivity can change, notably accompanied by temperature. By applying some suitable transformations, the partial differential governing equations turn down to the local partial differential formation, whose boundary conditions are not similar. The equations that are boundary layers are transformed and numerically solved, making use of the implicit finite difference technique with the Keller-box method [43]. We concentrate on the friction of local skin that is surface shear stress and Nusselt number that is local, in addition to the rate of transferring heat, the streamlines and the isotherms for $\gamma$, and the variation parameter of thermal conductivity.

## 2. Formulation of the Problem

Although the boundary layer analysis described below allows for various values of $\sigma(X)$, our elaborate numerical work made the assumption that the surface has sinusoidal

deformations. The surface that is wavy can be narrated by

$$Y_w = \sigma(X) = \alpha \sin\left(\frac{n\pi X}{L}\right), \tag{1}$$

where the characteristic length of the wave is $L$ which is related to the surface which is wavy.

The two-dimensional Cartesian coordinate system and the geometry of the wavy surface are exhibited in Figure 1 (see [30]).

We consider the flow under the usual approximation of Boussinesq governed by successive equations of boundary [2, 5, 10, 20, 27, 29, 30]:

Continuity equation:

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0. \tag{2}$$

Equation of $X$-momentum:

$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = -\frac{1}{\rho}\frac{\partial P}{\partial X} + v\nabla^2 U + g\beta(T - T_\infty). \tag{3}$$

Equation of $Y$-momentum:

$$U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y} = -\frac{1}{\rho}\frac{\partial P}{\partial Y} + v\nabla^2 V. \tag{4}$$

Energy equation:

$$U\frac{\partial T}{\partial X} + V\frac{\partial T}{\partial Y} = \frac{1}{\rho C_p}\nabla.(k\nabla T), \tag{5}$$

where dimensional coordinates $(X, Y)$ of the surface along with and normal to tangent $(U, V)$ are the components of velocity which are parallel of $X$ and $Y$, $k$ is fluid's thermal conductivity in the boundary zone which depends on fluid temperature, and $v(= \mu/\rho)$ is kinematics viscosity.

For this study, the boundary conditions are as follows [2, 5, 10, 20, 27, 29, 30]:

$$\begin{aligned} \text{At} \quad & T = Y \quad Y_w = \sigma(x), \quad T = T_w, U = V = 0. \\ \text{As} \quad & Y \longrightarrow \infty, \quad T = T_\infty, U = 0, \quad P = P_\infty. \end{aligned} \tag{6}$$

In the literature, a very few appearances of variations of thermal conductivity are available. Out of them, we here consider appropriate one for liquid obtained by Hossain et al. [19] as follows:

$$k = \frac{k_\infty}{[1 + \gamma^*(T - T_\infty)]}, \tag{7}$$

where ambient fluid's thermal conductivity is $k_\infty$ and $\gamma^*$ are constants determined at flow's film temperature $T_f = 1/2(T_w + T_\infty)$.

The governing partial differential equations are reduced to locally nonsimilar partial differential forms for solving
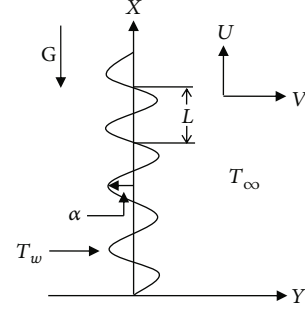


Figure 1: Physical model with coordinate system.

according to the mentioned method by adopting some appropriate transformations. From Yao [40], now we bring in the nondimensional variables given as follows:

$$\begin{aligned} x &= \frac{X}{L}, \\ x &= \frac{Y - \sigma}{L}, \\ p &= \frac{L^2}{\rho v^2}Gr^{-1}P, \\ u &= \frac{\rho L}{\mu_\infty}Gr^{-1/2}U, \\ v &= \frac{\rho L}{\mu_\infty}Gr^{-1/4}(V - \sigma_x U), \\ \theta &= \frac{T - T_\infty}{T_w - T_\infty}, \\ \sigma_x &= \frac{d\sigma}{dX} = \frac{d\sigma}{dx}, \\ Gr &= \frac{g\beta(T_w - T_\infty)}{v^2}L^3. \end{aligned} \tag{8}$$

Into equations (2)–(5) by introducing the above dimensionless variables, the obtained dimensionless governed equations are as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{9}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + Gr^{1/4}\sigma_x\frac{\partial p}{\partial y} + \left(1 + \sigma_x^2\right)\frac{\partial^2 u}{\partial y^2} + \theta, \tag{10}$$

$$\sigma_x\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -Gr^{1/4}\frac{\partial p}{\partial y} + \sigma_x\left(1 + \sigma_x^2\right)\frac{\partial^2 u}{\partial y^2} - \sigma_{xx}u^2, \tag{11}$$

$$u\frac{\partial\theta}{\partial x} + v\frac{\partial\theta}{\partial y} = \frac{1}{P_r}\frac{\left(1 + \sigma_x^2\right)}{(1 + \gamma\theta)}\frac{\partial^2\theta}{\partial y^2} - \frac{1}{P_r}\frac{\gamma\left(1 + \sigma_x^2\right)}{(1 + \gamma\theta)^2}\left(\frac{\partial\theta}{\partial y}\right)^2, \tag{12}$$

TABLE 1: Comparison of the present numerical results of skin friction coefficient, $f''(x, 0)$, and the rate of heat transfer, $-\theta'(x, 0)$, with Hossain et al. [20] for the variation of the Prandtl number Pr while $M = 0.0$, $\gamma = 0.0$, and $\varepsilon = 0.0$ with $\alpha = 0.1$.

| | $f''(x, 0)$ | | $-\theta'(x, 0)$ | |
| Pr | Hossain et al. [20] | Present work | Hossain et al. [20] | Present work |
| --- | --- | --- | --- | --- |
| 1.0 | 0.908 | 0.90814 | 0.401 | 0.40101 |
| 10.0 | 0.591 | 0.59269 | 0.825 | 0.82663 |
| 25.0 | 0.485 | 0.48733 | 1.066 | 1.06847 |
| 50.0 | 0.485 | 0.41727 | 1.066 | 1.28879 |
| 100.0 | 0.352 | 0.35559 | 1.542 | 1.54827 |

where $P_r = C_p \mu_\infty / k_\infty$ is the Prandtl number and $\gamma = \gamma^* (T_w - T_\infty)$ is the thermal conductivity variation parameter.

Equation (11) stipulates that along $y$-direction the pressure gradient is $O(Gr^{-1/4})$, which implies that from the inviscid flow solution along $x$-direction lowest order pressure gradient can be set on. This gradient of pressure is zero $(\partial p / \partial x = 0)$ for the present problem. Furthermore, equation (11) exhibits that $Gr^{1/4}(\partial p / \partial y)$ is $O(1)$ and with the help of the left-hand side of the equation, it is determined. The removal of $\partial p / \partial y$ from equation (10) to equation (11) thus induces to as follows:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \left(1 + \sigma_x^2\right) \frac{\partial^2 u}{\partial y^2} - \frac{\sigma_x \sigma_{xx}}{1 + \sigma_x^2} u^2 + \frac{1}{1 + \sigma_x^2} \theta. \quad (13)$$

For the present problem, the corresponding conditions of boundary are as follows:

$$\begin{aligned} \text{At} \quad y = 0, & \quad \theta = 1, \quad u = 0, \quad v = 0. \\ \text{As} \quad y \longrightarrow \infty, & \quad \theta = 0, \quad u = 0, \quad p = 0. \end{aligned} \right\} \quad (14)$$

We now initiate the conversions given below to bring down the governed equations to an opportune form as follows:

$$\begin{aligned} \eta &= y x^{-1/4}, \\ \psi &= x^{3/4} f(x, \eta), \\ \theta &= \theta(x, \eta), \end{aligned} \quad (15)$$

where $\psi$ is function of stream and $\eta$ is similarity variable of pseudo that satisfies equation (9).

Into equations (13) and (12), bringing the transformations which gives in equation (15), the system of nonlinear equations obtained is presented as follows:

$$\left(1 + \sigma_x^2\right) f''' + \frac{3}{4} f f'' - \left(\frac{1}{2} + \frac{x \sigma_x \sigma_{xx}}{1 + \sigma_x^2}\right) f'^2 + \frac{1}{1 + \sigma_x^2} \theta = x \left(f' \frac{\partial f'}{\partial x} - f'' \frac{\partial f}{\partial x}\right),$$

$$\frac{1}{P_r} \frac{\left(1 + \sigma_x^2\right)}{1 + \gamma \theta} \theta'' - \frac{1}{P_r} \frac{\gamma\left(1 + \sigma_x^2\right)}{\left(1 + \gamma \theta\right)^2} \theta'^2 + \frac{3}{4} f \theta' = x \left(f' \frac{\partial \theta}{\partial x} - \theta' \frac{\partial f}{\partial x}\right). \quad (16)$$

Now the conditions of boundary (14) become the form below:

$$\begin{aligned} \theta(x, 0) &= 1, \quad f(x, 0) = f'(x, 0) = 0, \\ \theta(x, \infty) &= 0, \quad f'(x, \infty) = 0. \end{aligned} \right\} \quad (17)$$

In practical applications, the physical quantities of principle interest are the shearing stress $\tau_w$ and the rate of heat transfer in terms of the skin friction coefficient $C_{fx}$ and Nusselt number $Nu_x$, respectively, which can be written as follows:

$$\begin{aligned} C_{fx} &= \frac{2 \tau_w}{\rho U^2}, \\ Nu_x &= \frac{q_w x}{k_\infty (T_w - T_\infty)}, \end{aligned} \quad (18)$$

where

$$\begin{aligned} q_w &= -k (n.\nabla T)_{y=0}, \\ \tau_w &= (\mu n.\nabla U)_{y=0}. \end{aligned} \quad (19)$$

Using the transformations (15) and (19) into (18), $C_{fx}$, the friction coefficient of the skin and the Nusselt number where both are local that is the rate of heat transfer $Nu_x$, takes the form below:

$$\begin{aligned} C_{fx} \left(\frac{Gr}{x}\right)^{-1/4} &= \sqrt{1 + \sigma_x^2} f''(x, o), \\ Nu_x \left(\frac{Gr}{x}\right)^{-1/4} &= -\frac{\sqrt{1 + \sigma_x^2}}{(1 + \gamma)} \theta'(x, o). \end{aligned} \quad (20)$$

## 3. Methodology of the Solution

The study discusses with the incompressible fluid's free convective flow that is viscous along a wavy vertical surface and is heated; uniform and variable thermal conductivity inversely proportional to temperature, which is linear function, has been investigated by utilizing the implicit finite difference technique familiar as the Keller-box method, which was introduced by Keller [43]. Recently, this technique has extensively applied by Hossain et al. [19, 20]. Here, in the implicit finite difference method (IFDM), we introduce new dependent variables $u(\xi, \eta), v(\xi, \eta), p(\xi, \eta)$, and $g(\xi, \eta)$ to transform momentum and energy equations, where $x = \xi$ and $\theta = g$ are used for boundary conditions and some suitable values are used for nondimensional variables. Then, consider the net rectangle on the $(\xi, \eta)$ plane for the box scheme. After that, the transformed momentum and energy equations are calculated by the finite difference approximations, approximating the functions and their derivatives in terms of the central difference approximations according to box method. The above central difference approximations reduce the system of first order differential equations to a set of nonlinear difference equations for the unknown at $x_i$ in terms of their values at
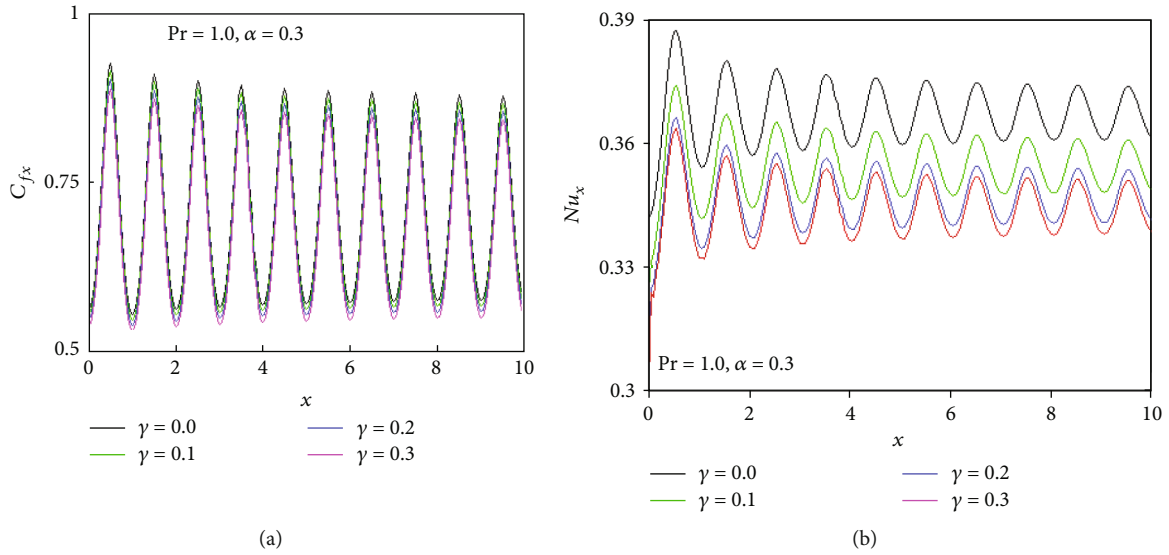
FIGURE 2: Variety of (a) $C_{fx}$, frictional coefficient of skin, and (b) $Nu_x$, heat transferring rate against $x$, the distance that is dimensionless for various thermal conductivity values while Pr = 1.0 and $\alpha = 0.3$.
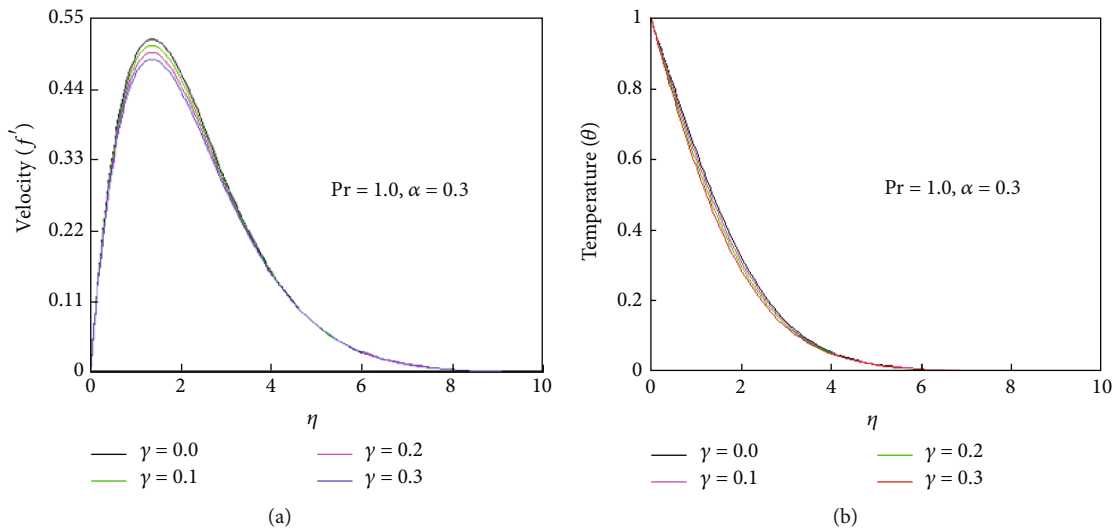


FIGURE 3: Variation of (a) $f'$, the velocity profiles, and (b) $\theta$, the dispensation of temperature in oppose to the distance $\eta$ that is dimensionless for distinct values of $\gamma$, the variation variable of thermal conductivity while Pr = 1.0 and $\alpha = 0.3$.

$x_{i-1}$. The resulting nonlinear systems of algebraic equations are to be linearized by Newton's Quassy linearization method. Iterating higher and substituting, we got the algebraic form of the linear system. Calculating and having coefficient of momentum and energy equations and boundary conditions, the system of linear equations together with the boundary conditions can be written in matrix or vector form, where the coefficient matrix has a block tridiagonal structure. The whole procedure, namely, reduction to first order followed by central difference approximations, Newton's quasilinearization method, and the block Thomas algorithm, is well known as the Keller-box method.

## 4. Validation of the Code

A comparison of the present numerical results of skin friction coefficient, $f''(x, 0)$, and the rate of heat transfer, $-\theta'(x, 0)$, with the results obtained by Hossain et al. [20] is depicted in Table 1. Here, the magnetic parameter $M$, thermal conductivity parameter $\gamma$, and viscosity parameter $\varepsilon$ were ignored while different values of the Prandtl number Pr = $(1.0, 10.0, 25.0, 50.0, \text{and } 100.0)$ are chosen with amplitude-to-length ratio $\alpha = 0.1$. From Table 1, it is clearly seen that the present results are in excellent agreement with the solution of Hossain et al. [20].
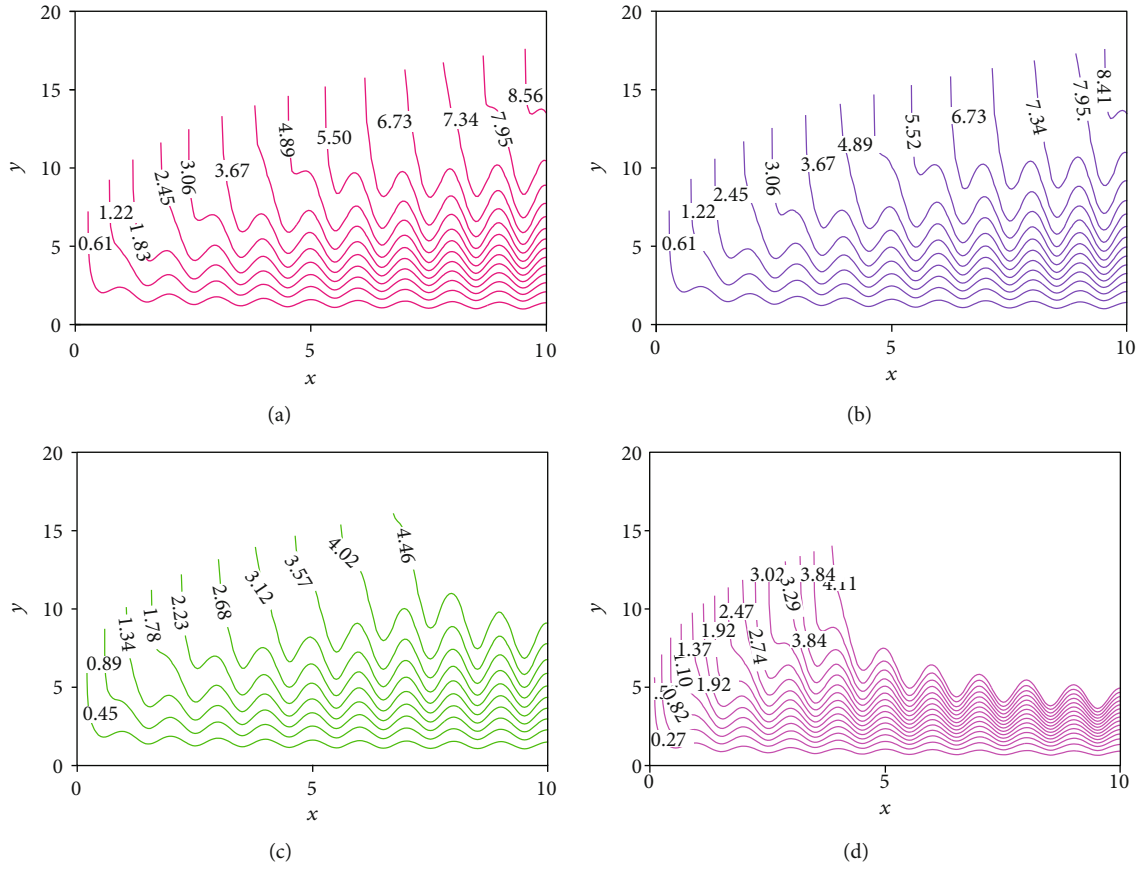
(a)

(b)

(c)

(d)

FIGURE 4: Effects of streamlines while Pr = 1.0 and $\alpha$ = 0.3, for (a) $\gamma$ = 0.0, (b) $\gamma$ = 0.1, (c) $\gamma$ = 0.2, and (d) $\gamma$ = 0.3.

## 5. Results and Discussion

The heat transfer rate regarding $\text{Nu}_x$, the Nusselt number, the numeric values of $C_{fx}$, the friction coefficient of skin, the isotherms and streamlines, temperature, and velocity profiles are obtained and presented graphically for $\gamma$, various values of thermal conductivity varying variables ranging from 0 (the thermal conductivity that is constant) to 0.3 while other controlling parameters, the amplitude-to-length ratio $\alpha$ = 0.3 and the Prandtl number Pr = 1.0, and for various Pr, the Prandtl number = 0.73, 1.73, 3.0, 7.0, and 15.5 with $\alpha$ = 0.3 and $\gamma$ = 0.05. Here, Pr = 0.73, 1.73, 3.0, 7.00, and 15.5 correspondent, respectively, at 2100°K to the air and at 100°C, 60°C, and 20°C to water and $CaCl_2$. In this study, the values of various parameters are chosen according to analyzing other work and also taking suitability into account. Some values are chosen up to the point at which they behave normally. After that value, that is some more or less value of that parameter, the flow behavior loses its originality.

Figures 2(a) and 2(b) demonstrate, respectively, the changes of temperature-dependent skin friction coefficient and heat transfer rate for the reciprocal variety of thermal conductivity ($\gamma$ = 0, 0.1, 0.2, and 0.3) when Pr = 1.0 and $\alpha$ = 0.3. We know that the higher value of thermal conductivity ($\gamma$) accelerates the fluid flow and so increases $C_{fx}$ and

$\text{Nu}_x$. Here, as $\gamma$ is reciprocal, so the flow behavior will also reverse its normal attitude. For this reason, when rising $\gamma$, the frictional coefficient of skin and the heat transferring rate notably decrease along with direction of downstream, the surface on the way to direction of $x$ axial. The highest values of $C_{fx}$, the frictional coefficient of skin that are local, are found as 0.92568, 0.91336, 0.90034, and 0.88662 at $x = 0.50$ for $\gamma = 0$, 0.1, 0.2, and 0.3, respectively. Nonetheless, the maximum values of the heat transfer rate at $x = 0.55$ are 0.38755 for =0.0, 0.37409 for =0.1, 0.36637 for =0.2, and 0.36369 for =0.3. When the difference between them is from 0.0 to 0.3, the heat transfer rate and skin friction coefficient decrease by about 6% and 4%, respectively. The high thermal conductivity speeds up fluid flow that raises the heat transfer coefficient and, additionally, the frictional coefficient of skin.

The variation of velocity profile and temperature profile within the boundary layer for the reciprocal variation of thermal conductivity ($\gamma$ = 0, 0.1, 0.2, and 0.3) with temperature dependent is shown in Figures 3(a) and 3(b), respectively, when Pr = 1.0 and $\alpha$ = 0.3. As $\gamma = \gamma^*(T_w - T_\infty)$ and presently the variation of thermal conductivity is conversely proportional, the rising value of thermal conductivity decays temperature variation between the temperature of the surface and the fluid's ambient temperature. Heat is transferred then slowly within the boundary layer from the surface to
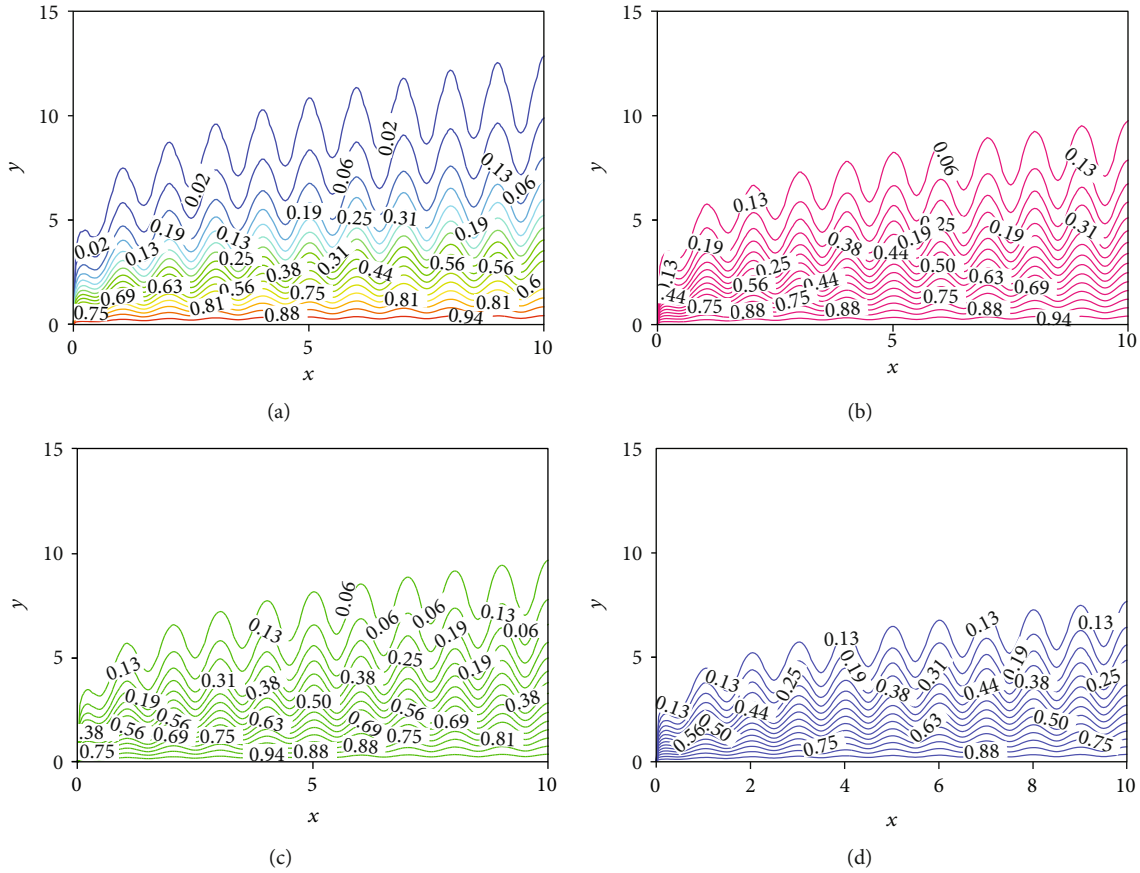
FIGURE 5: Effect of isotherms for (a) $\gamma = 0.0$, (b) $\gamma = 0.1$, (c) $\gamma = 0.2$, and (d) $\gamma = 0.3$ while Pr = 1.0 and $\alpha = 0.3$.

the fluid. With the increasing value of thermal conductivity, both temperature and velocity decrease monotonically. Additionally, the biggest velocities are stored as 0.51730, 0.50709, 0.49662, and 0.48594 for $\gamma = 0$, 0.1, 0.2, and 0.3, respectively, and every one of them happens at the similar spot of $\eta$. It is found that velocity decays approximately by 6% as $\gamma$ changes from 0 to 0.3.

On the formation of streamlines and isotherms, Figures 4 and 5 show the effects of thermal conductivity, which is temperature dependent and whose variation is reciprocal with different stable variables Pr = 1.0 and $\alpha = 0.3$. For $\gamma = 0.0$, 0.1, 0.2, and 0.3, it can be seen that the maximum values of $\psi$ and $\psi_{max}$ are 8.56, 8.41, 4.46, and 4.11, respectively. Moreover, increasing thermal conductivity while varying parameters decreases the boundary layer. Originally, rising $\gamma$ increases both streamline and isotherm. But for reciprocal changes, the behavior also goes through reverse.

The effects of various Prandtl numbers (Pr) are equal to 0.73, 1.73, 3.0, 7.0, and 15.5 while $\gamma = 0.05$ and $\alpha = 0.3$ on friction coefficient of skin, and heat transfer rate has been illustrated, respectively, in Figures 6(a) and 6(b). The skin friction coefficient and the heat transfer rate decrease as the Prandtl number increases. The rising values of the Prandtl number hasten the decay of the temperature from the surface which is warmed with subse-

quent growth in the heat transferring rate. The highest local skin friction coefficients are observed as 0.963088 and 0.55011, and the local heat transfer rates are 0.34039 and 0.88817, respectively, for Pr = 0.73 and 15.5. It is clear that the local skin frictional coefficient decays approximately by 43%, and the heat transfer rate increases approximately by 61% as the Prandtl number rises from 0.73 to 15.5.

Figure 7(a) depicts the interactivity of the Prandtl number on velocity $f'(x, \eta)$, while Figure 7(b) depicts the temperature $\theta(x, \eta)$, with different effecting variables $\gamma = 0.05$ and $\alpha = 0.3$. The Prandtl number is the quotient of two forces. Those are the viscous and thermal forces. Rising Pr increases viscosity and decays the thermal action of fluids. Fluids cannot move freely if viscosity increases. Figure 7(a) shows that as the Prandtl number increases, the velocity of the fluid decreases rapidly along with the downward direction of the plate against $\eta$.

It is found that when Pr increases, velocity decreases by 64% because the biggest velocity is 0.55738 for Pr = 0.73 and 0.19935 for Pr = 15.5. It is clear from Figure 7(b) that as Pr increases, the Prandtl number temperature decreases. Generally, for increasing the Prandtl number (Pr), both momentum, that is, streamline and the boundary layer, that is, isotherm, become thinner. With other controlling
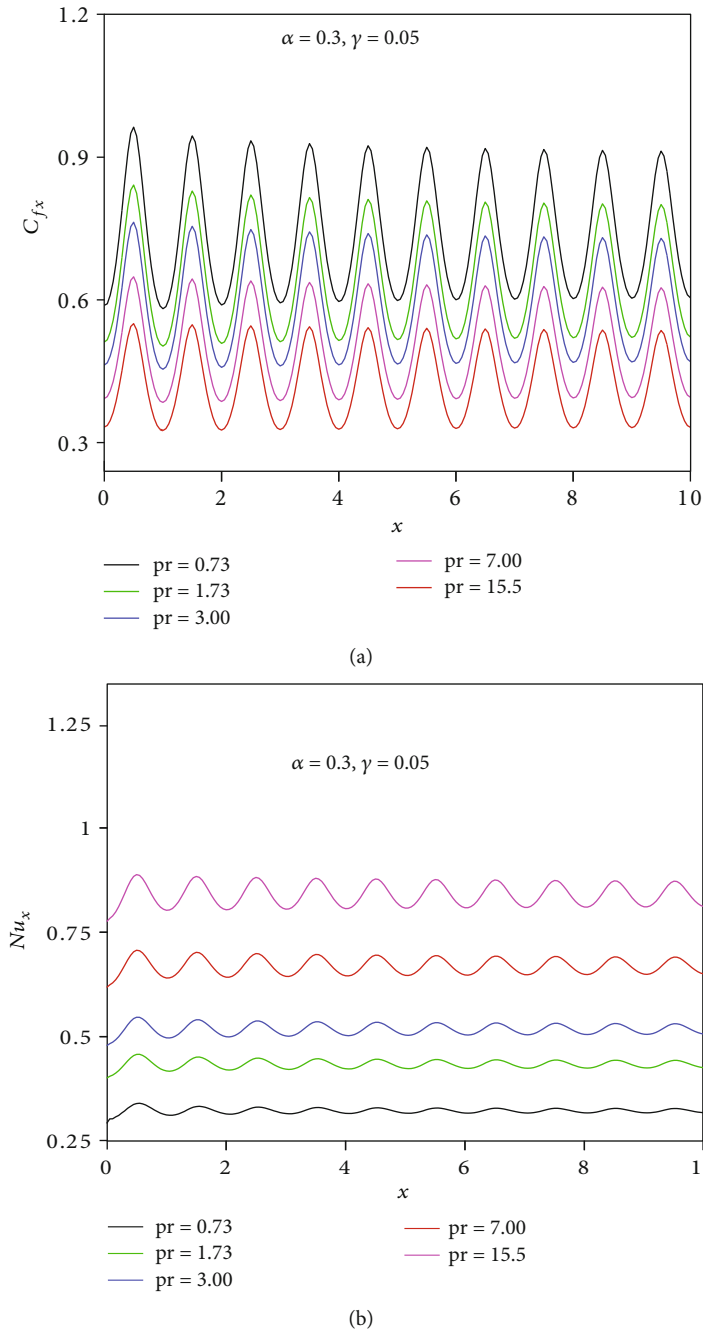
(a)



(b)

FIGURE 6: Variety of (a) skin friction coefficient ($C_{fx}$) and (b) heat transferring rate ($\mathrm{Nu}_x$) against distance $x$ that is dimensionless for various Prandtl numbers while $\gamma = 0.05$ and $\alpha = 0.3$ (see [30]).

parameters $\gamma = 0.05$ and $\alpha = 0.3$, the transformation of the variety of Prandtl numbers over the forms of the streamlines and the isotherms is shown in Figures 8 and 9. It can be observed that for Pr = 0.73, 1.73, 3.00, and 7.00, the highest $\psi$, that is, $\psi_{\max}$, are 9.17, 6.73, 5.35, and 4.11, respectively. So, it can be included that for the lower Prandtl number with the effects of thermal conductivity, that is, temperature dependent and additionally amplitude-to-length ratio, the boundary layer and momentum are set off thinner.

The study of free convective heat transfer from an irregular surface has drawn the attention of scientists and engineers because natural convective heat transfer is important in many natural and industrial problems, and transferring heat from an irregular surface is used in many heat transferring devices. It is possible to conclude from the results that the work has a significant impact, so it will be very noble, going above and beyond previous efforts, that this thesis will be useful for engineers in geophysics and energy-related engineering, industrial applications, heat transfer devices, and so on.
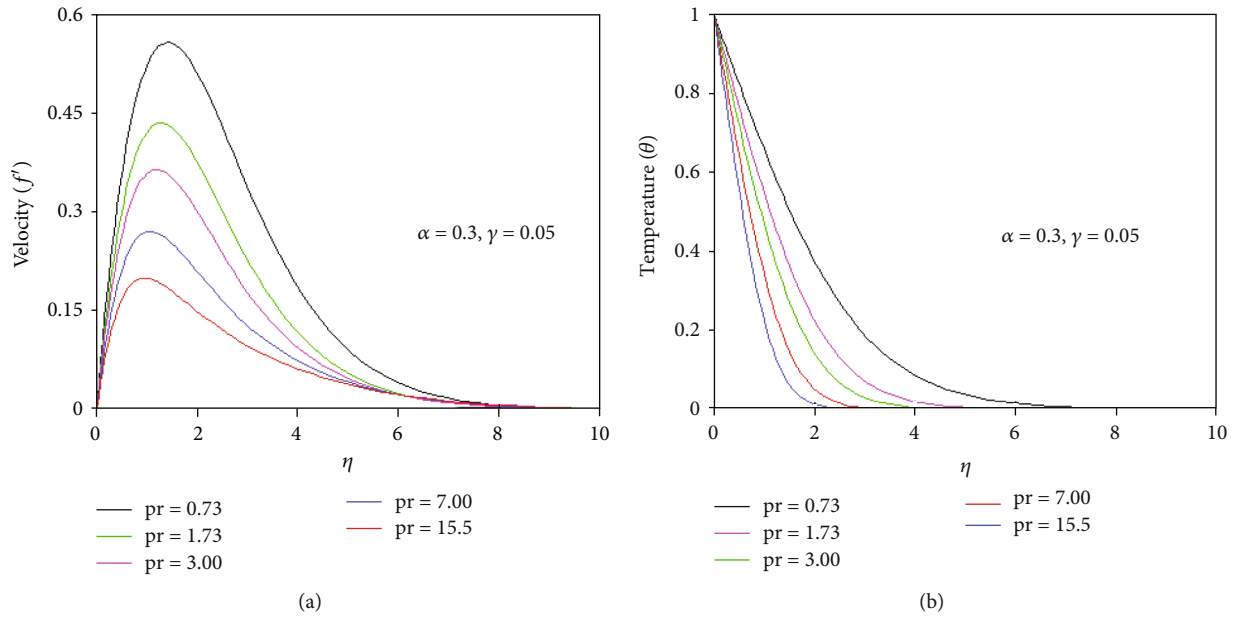
FIGURE 7: Variation of (a) the profiles of velocity $(f')$ and (b) the dispensation of temperature $(\theta)$ for various Prandtl numbers, Pr against dimensionless distance $\eta$ while $\gamma = 0.05$ and $\alpha = 0.3$.
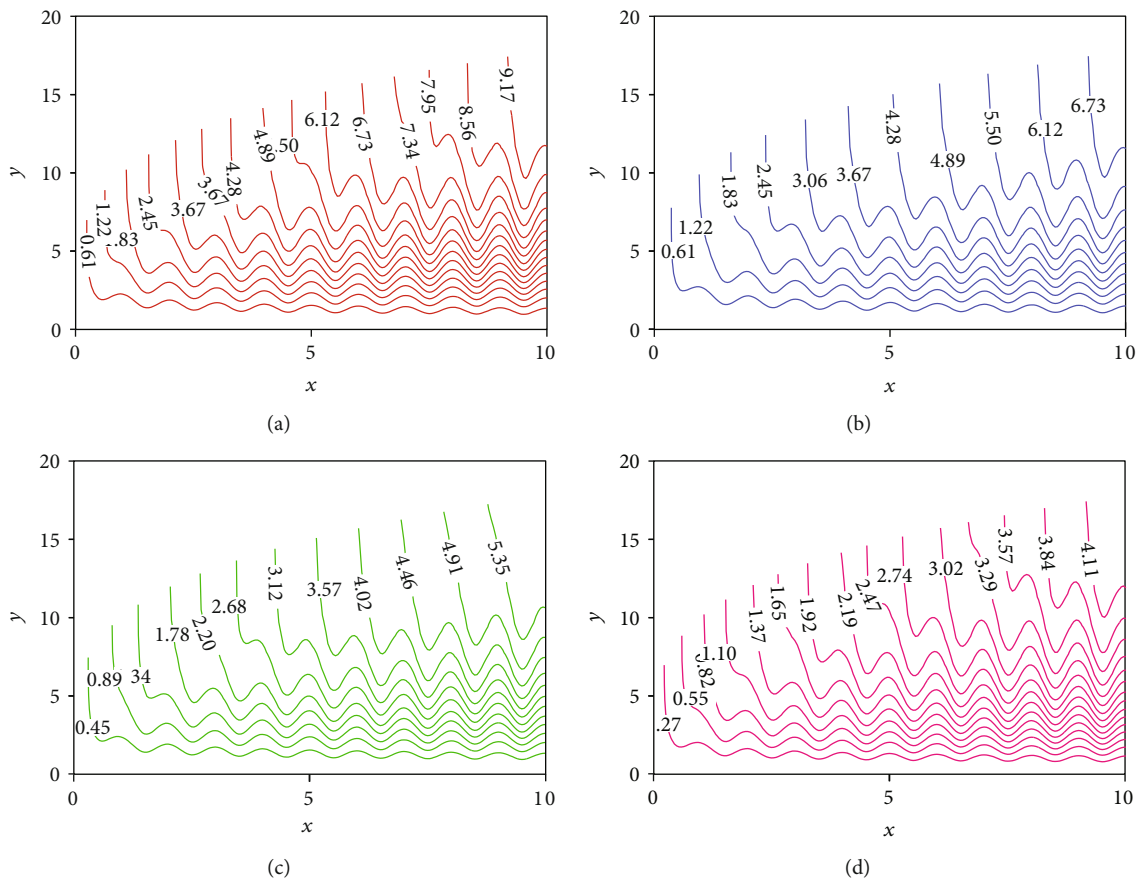


FIGURE 8: Effect of streamlines for (a) Pr = 0.73, (b) Pr = 1.73, (c) Pr = 3.0, and (d) Pr = 7.0 when $\alpha = 0.3$ and $\gamma = 0.05$.
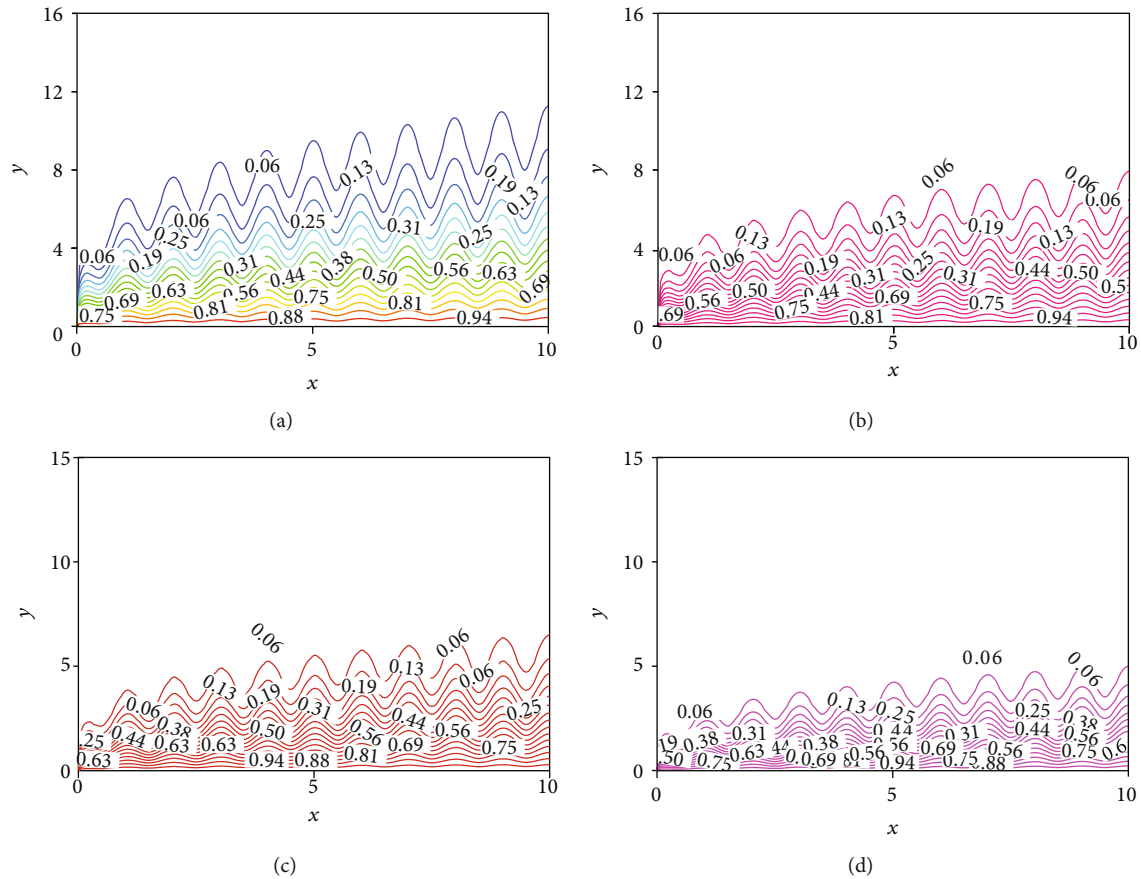
(a)

(b)

(c)

(d)

FIGURE 9: Effect of isotherms while $\gamma = 0.05$ and $\alpha = 0.3$, for (a) Pr = 0.73, (b) Pr = 1.73, (c) Pr = 3.0, and (d) Pr = 7.0.

## 6. Conclusion

The concept of natural convective heat transport of incompressible viscous fluids with a reciprocal variety of the thermal conductivity, which is temperature dependent along a surface, that is, wavy and vertical, has been studied. This work explored the impact of a range of thermal conductivities that are temperature dependent and conversely proportional with a linear function of temperature on free convective flow over a heated evenly and wavy vertical surface where the fluids are viscous and incompressible. The governing equations with adequate boundary conditions are converted into nondimensional form and analyzed using the implicit central finite difference methodology with Newton's quasilinearization method and the block Thomas algorithm, also known as the Keller-box method. The results in terms of heat transfer rate, skin frictional coefficient, isotherms, and streamlines are graphically displayed for the effects of various physical parameters. The following are short outlines of the important outcomes:

(i) It is found that the heat transferring rate is local, the frictional coefficient of the skin is local, and the temperature and the velocity are decreasing with the rising values of the reciprocal variation of thermal conductivity, which is temperature dependent

(ii) It is remarked that heat transfer rate and skin friction coefficient decrease approximately 6% and 4% when $\gamma$ differs from 0.0 to 0.3, respectively

(iii) Both the thermal boundary layer and the velocity boundary layer become narrower when the effect of thermal conductivity, which is temperature dependent, is considered

(iv) It is found that velocity decays approximately by 6% as $\gamma$ changes 0.0 to 0.3

(v) The friction coefficient of skin, the velocity, and the temperature decrease, but the heat transfer rate increases as Prandtl's number increases. Here, the friction coefficient of skin and velocity decrease by 43% and 64%, respectively, while the heat transfer rate rises by approximately 61%.

(vi) With the effects of the Prandtl number, both the boundary layer thermal and velocity become thinner

## Data Availability

All the data are available in manuscripts.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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