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Research Article

Stationary Solutions to the Three-Dimensional Compressible Nonisothermal Nematic Liquid Crystal Flows

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In this paper, we study the stationary compressible nonisothermal nematic liquid crystal flows affected by the external force of general form in three-dimensional space. By using the contraction mapping principle, we prove the existence and uniqueness of strong solution around the constant state in some suitable function space.

1. Introduction and Main Result

We consider the problem of identifying the stationary motion of compressible nonisothermal nematic liquid crystal flows effected by the external force of general form:

$$\begin{cases} \operatorname{div}(\rho u) = H, \\ \rho(u\cdot\nabla)u - \mu\Delta u - (\mu+\lambda)\nabla \operatorname{div} u + \nabla P = -\nabla d\cdot\Delta d + \rho F, \\ \rho(u\cdot\nabla)\theta + P \operatorname{div} u - \kappa\Delta\theta = \Psi(u) + \left|\Delta d + |\nabla d|^2 d\right|^2 + G, \\ (u\cdot\nabla)d - \Delta d = |\nabla d|^2 d + R. \end{cases} \tag{1}$$

Here, the unknown functions $\rho = \rho(x) > 0$, $u = u(x) \in \mathbb{R}^3$, $\theta = \theta(x) > 0$, and $d = d(x) \in \mathbb{S}^2$ are the density, velocity, absolute temperature, and macroscopic average of the nematic liquid crystal orientation field, respectively. The pressure $P = P(\rho,\theta) > 0$ is a smooth function of ρ, θ satisfying $P_\rho(\rho,\theta) > 0$, $P_\theta(\rho,\theta) > 0$. The constants μ, λ are the shear and bulk viscosity coefficients of the fluids, respectively, which are assumed to satisfy the physical restrictions $\mu > 0$ and $2\mu + 3\lambda \geq 0$. The constant $\kappa > 0$ is the ratio of the heat conductivity coefficient over the heat capacity. Moreover, $\Psi = \Psi(u)$ is the dissipation function:

$$\Psi(u) = \frac{\mu}{2}\left|\nabla u + (\nabla u)^T\right|^2 + \lambda(\operatorname{div}\, u)^2. \tag{2}$$

In addition, $H(x)$, $F(x) = (F_1(x), F_2(x), F_3(x))$, $R(x) = (R_1(x), R_2(x), R_3(x))$, and $G(x)$ are the given mass source, external force, director source, and energy source, respectively, which are given by the following form

$$(H, F, G, R) = \operatorname{div}(H_1, F_1, G_1, R_1) + (H_2, F_2, G_2, R_2), \tag{3}$$

where $H_1 = (H_1^i(x))_{1\leq i\leq 3}$, $H_2 = H_2(x)$, $F_1 = (F_1^{ij}(x))_{1\leq i,j\leq 3}$, $F_2 = (F_2^i(x))_{1\leq i\leq 3}$, $G_1 = (G_1^i(x))_{1\leq i\leq 3}$, $G_2 = G_2(x)$, and $R_1 = (R_1^{ij}(x))_{1\leq i,j\leq 3}$, $R_2 = (R_2^i(x))_{1\leq i\leq 3}$.

As the space variable tends to infinity, we assume

$$(\rho, u, \theta, d)(x) \longrightarrow \left(\bar\rho, 0, \bar\theta, \bar d\right) \text{ as } |x| \longrightarrow +\infty, \tag{4}$$

where $\bar\rho > 0$ and $\bar\theta > 0$ are two given constants, and $\bar d$ is a unit constant vector.

The flow of nematic liquid crystals can be regarded as slow moving particles, in which the fluid velocity and the particles arrangement influence each other. The continuum theory of the nematic liquid crystals was first proposed by Ericksen [1] and Leslie [2] during the period between 1958 and 1968, see also the book by De Gennes [3]. The hydrodynamic flow of nematic liquid crystal system has attracted great interest and generated many important developments. Here, we only mention some of related to our study. When the temperature is absent, i.e., $\theta = 0$. In [4, 5], Huang et al. addressed some issues on the strong solutions in three-dimensional space. Specifically, in [4], the authors established a blow-up criterion, while the local existence of a unique strong solution if the initial data are sufficiently regular and satisfy a natural compatibility condition are studied in [5]. Later, Jiang et al. [6] considered the global existence of weak solutions with large initial energy and without any smallness condition on the initial density and velocity in a bounded domain of the multidimensional space. By using the domain expansion technique and the rigidity theorem, these authors [7] also proved the global existence of large weak solutions in two dimensions, provided that the second component of initial data of the direction field satisfies some geometric angle condition. One may also see [8] for some recent progress on the existence, regularity, uniqueness, and large time asymptotic of the nematic liquid crystal flows.

Recently, Feireisl et al. and Feireisl et al. [9, 10] established nonisothermal models of incompressible nematic liquid crystals and obtained the global existence of weak solutions. For further study about the well-posedness of solutions to the incompressible nematic liquid crystal flows, we refer to [11–15] and references therein. For the compressible nonisothermal nematic liquid crystal flows, Guo et al. [16] shown the existence of global weak solutions by a three-level approximation and weak convergence when the adiabatic exponent $\gamma > 3/2$. Moreover, they [17] also considered the global existence and decay rates of small smooth solutions in the whole space $\mathbb{R}^3$. Later, Fan et al. [18] proved the local well-posedness of strong solutions to the initial boundary value problem in $3D$. By some delicate energy estimates and the critical Sobolev inequalities of logarithmic type, Zhong [19] obtained the singularity formation of strong solutions to the initial boundary problem in $2D$. Then, Liu and Zhong [20] established the global existence and uniqueness of strong solutions with vacuum as far field density in three-dimensional space. However, a lot of physical and mathematical important problems are still open due to the lack of a smoothing mechanism and the strong nonlinearity. Up to now, there are no result available on the existence of stationary solutions to the three-dimensional compressible nonisothermal nematic liquid crystal flows with external force.

Inspired by the work of [21, 22] for the compressible Navier-Stokes equations, the aim of this paper is to study the existence and uniqueness of stationary solution, which is a small strong solution around the constant state $(\bar{\rho}, 0, \bar{\theta}, \bar{d})$ to problem (1)–(4). It is worth mentioning here that the nonisothermal nematic liquid crystal flow system (1) adds the strong nonlinear terms, which seems more complicated than that of the Navier-Stokes equations, and we should carefully deal with the direction field of liquid crystals in the angular momentum equation. Moreover, we think that discussing the stationary solutions to (1) is of significance to some extent.

As in [21], we choose (P, u, θ, d) as the independent variables and regard ρ as a smooth function of (P, θ). To this end, set $\bar{P} = P(\bar{\rho}, \bar{\theta})$ and denote

$$\varrho = P - \bar{P}, \vartheta = \theta - \bar{\theta}, w = d - \bar{d}. \tag{5}$$

Then, (1) can be reformulated as

$$\begin{cases} \operatorname{div} u + \dfrac{\rho_P}{\rho}(u \cdot \nabla)\varrho = -\dfrac{\rho_\theta}{\rho}(u \cdot \nabla)\vartheta + \dfrac{H}{\rho}, \\ -\mu\Delta u - (\mu + \lambda)\nabla \operatorname{div} u + \nabla\varrho = -\rho(u \cdot \nabla)u - \nabla w \cdot \Delta w + \rho F, \\ -\kappa\Delta\vartheta = -\left(\rho + \dfrac{\rho_\theta}{\rho}\left(\varrho + \bar{P}\right)\right)(u \cdot \nabla)\vartheta - \dfrac{\rho_P}{\rho}\left(\varrho + \bar{P}\right)(u \cdot \nabla)\varrho + \Psi(u) + \left|\Delta w + |\nabla w|^2\left(w + \bar{d}\right)\right|^2 + \left(\varrho + \bar{P}\right)\dfrac{H}{\rho} + G, \\ -\Delta w = -(u \cdot \nabla)w + |\nabla w|^2\left(w + \bar{d}\right) + R. \end{cases} \tag{6}$$

As a preparation for stating our main result, we introduce some notations and conventions to be used throughout this paper. Here, C denotes a generic positive constant which may vary depending on the estimate. $\nabla^l f$ with any integer $l \geq 0$ stands for all derivatives up to l-th order of the function f with respect to the spatial variable x. $L^p(\mathbb{R}^3), 1 \leq p \leq \infty$ stands for the usual L^p spaces with norm $\|\cdot\|_{L^p}$ and for any integer $m \geq 0$, $H^m(\mathbb{R}^3)$ stands for the usual L^2-Sobolev spaces with norm $\|\cdot\|_{H^m}$. Let

$$\widehat{H}^m(\mathbb{R}^3) = \left\{u \in L^1_{loc}(\mathbb{R}^3) : \nabla u \in H^{m-1}(\mathbb{R}^3)\right\}. \tag{7}$$

Finally, we define some function spaces. For any $\varepsilon > 0$, denote

$$M^k_\varepsilon = \left\{\varrho : \|\varrho\|_{M^k} \leq \varepsilon\right\}, V^k_\varepsilon = \left\{v : \|v\|_{V^k} \leq \varepsilon\right\}, \tag{8}$$

with

$$\begin{aligned}
\|\varrho\|_{M^k} &= \|\varrho\|_{L^6} + \left\|(1+|x|)^2\varrho\right\|_{L^\infty} + \sum_{\nu=1}^{k}\|(1+|x|)^\nu\nabla^\nu\varrho\|_{L^2},\\
\|v\|_{V^k} &= \|v\|_{L^6} + \sum_{\nu=0}^{1}\left\|(1+|x|)^{\nu+1}\nabla^\nu v\right\|_{L^\infty} + \sum_{\nu=1}^{k}\left\|(1+|x|)^{\nu-1}\nabla^\nu v\right\|_{L^2}.
\end{aligned} \tag{9}$$

Moreover, we put

$$\begin{aligned}
X_\varepsilon^{l_1,l_2,l_3,l_4} &= \left\{(\varrho,u,\vartheta,w)\colon \varrho\in M_\varepsilon^{l_1}, u\in V_\varepsilon^{l_2}, \vartheta\in V_\varepsilon^{l_3}, w\in V_\varepsilon^{l_4}, \|(\varrho,u,\vartheta,w)\|_{X^{l_1,l_2,l_3,l_4}}\le\varepsilon\right\},\\
\tilde{X}_\varepsilon^{l_1,l_2,l_3,l_4} &= \left\{(\varrho,u,\vartheta,w)\in X_\varepsilon^{l_1,l_2,l_3,l_4} : \operatorname{div} u = \operatorname{div} U_1 + U_2 \text{ for some } U_1, U_2 \text{ such that} \left\|(1+|x|)^3U_1\right\|_{L^\infty} + \left\|(1+|x|)^{-1}U_2\right\|_{L^1}\le\varepsilon\right\},\\
\mathcal{S} &= \left\{\varpi : \varpi = \operatorname{div}\varpi_1 + \varpi_2 \text{ for some } \varpi_1, \varpi_2 \text{ and satisfies } \|\varpi\|_{\mathcal{S}}<\infty\right\},
\end{aligned} \tag{10}$$

where

$$\|(\varrho,u,\vartheta,w)\|_{X^{l_1,l_2,l_3,l_4}} = \|\varrho\|_{M^{l_1}} + \|u\|_{V^{l_2}} + \|\vartheta\|_{V^{l_3}} + \|w\|_{V^{l_4}},$$

$$\begin{aligned}
\|\varpi\|_{\mathcal{S}} = &\sum_{\nu=0}^{3}\left\|(1+|x|)^{\nu+1}\nabla^\nu\varpi\right\|_{L^2} + \left\|(1+|x|)^3\varpi\right\|_{L^\infty}\\
&+ \left\|(1+|x|)^2\varpi_1\right\|_{L^\infty} + \|\varpi_2\|_{L^1}.
\end{aligned} \tag{11}$$

Now, we state our main result about the existence and uniqueness of stationary solutions (P,u,θ,d) to the stationary problem (1).

Theorem 1. *There exist small constants $\varepsilon_0>0$ and $c_0>0$ depending only on $\bar{\rho},\bar{\theta},\bar{d}$, such that if*

$$\|(H,F,G,R)\|_{\mathcal{S}} + \left\|(1+|x|)^4\nabla^4H\right\|_{L^2} + \left\|(1+|x|)^{-1}H\right\|_{L^1}\le c_0\varepsilon, \tag{12}$$

for some positive constant $\varepsilon\le\varepsilon_0$, then the problem (6) admits a solution $(\varrho,u,\vartheta,w)\in\tilde{X}_\varepsilon^{4,5,5,5}$.

Moreover, the solution is unique in the following sense: if there is another solution $(\varrho^1,u^1,\vartheta^1,w^1)$ satisfying (6) with the same (H,F,G,R), and $\|(\varrho^1,u^1,\vartheta^1,w^1)\|_{X^{4,5,5,5}}\le\varepsilon$, then $(\varrho^1,u^1,\vartheta^1,w^1)=(\varrho,u,\vartheta,w)$.

The rest of this paper is organized as follows. In Section 2, we establish the weighted L^2 theory for the corresponding linearized problem of (6). The proof for the existence and uniqueness of the stationary solutions to (6) will be considered in Section 3.

2. Weighted L^2 Theory for Linearized Problem

In this section, we study the weighted L^2 theory for the following linearized system of (6):

$$\begin{cases}
\operatorname{div} u + (a\cdot\nabla)\varrho = h,\\
-\mu\Delta u - (\mu+\lambda)\nabla\operatorname{div} u + \nabla\varrho = f,\\
-\kappa\Delta\vartheta = g,\\
-\Delta w = r.
\end{cases} \tag{13}$$

Here, $a=(a^1(x),a^2(x),a^3(x))$ and $(h,f,g,r)\in H^k\times H^{k-1}\times H^{k-1}\times H^{k-1}$ are given as

$$f=-(b_1\cdot\nabla)c_1+\tilde{f},\ g=-(b_2\cdot\nabla)c_2+\tilde{g},\ r=-(b_3\cdot\nabla)c_3+\tilde{r}. \tag{14}$$

Moreover, we fix $k=3$ or $k=4$ and always assume that

$$\begin{aligned}
a\in\hat{H}^4,\ &\sum_{\nu=1}^{4}\left\|(1+|x|)^{\nu-1}\nabla^\nu a\right\|_{L^2} + \|(1+|x|)a\|_{L^\infty}\\
&\le\eta, b_1, c_1, b_2, c_2, b_3, c_3\in V^{k+1},
\end{aligned} \tag{15}$$

$$\begin{aligned}
&\|(1+x)h\|_{L^2} + \sum_{\nu=1}^{k}\|(1+|x|)^\nu\nabla^\nu h\|_{L^2}\\
&+ \sum_{\nu=0}^{k-1}\left\|(1+|x|)^{\nu+1}\nabla^\nu\left(\tilde{f},\tilde{g},\tilde{r}\right)\right\|_{L^2}<\infty.
\end{aligned} \tag{16}$$

We would like to point out that equations $(13)_1$–$(13)_3$, which are independent of w, have been well studied in [21], while $(13)_4$ is easy to be handled. Hence, by using the same arguments as in [21], that is by using the Banach closed range theorem and some weighted–L^2 estimates on the linearized problem (13), we get the following result.

Theorem 2. *There exists a positive constants $\eta_0 > 0$ depending only on μ, λ and κ such that if η in (15) satisfies $\eta \le \eta_0$, then the problem (13) has a solution $(\varrho, u, \vartheta, w)$ satisfying*

$$\begin{aligned}
&\|(\varrho,u,\vartheta,w)\|_{L^6}+\sum_{\nu=1}^{k}\|(1+|x|)^{\nu}\nabla^{\nu}\varrho\|_{L^2}\\
&\quad+\sum_{\nu=1}^{k+1}\left\|(1+|x|)^{\nu-1}(\nabla^{\nu}u,\nabla^{\nu}\vartheta,\nabla^{\nu}w)\right\|_{L^2}\\
&\le C\Bigg\{\|(b_1,b_2,b_3)\|_{V^{k+1}}\|(c_1,c_2,c_3)\|_{V^{k+1}}+\|(1+|x|)h\|_{L^2}\\
&\quad+\sum_{\nu=1}^{k}\|(1+|x|)^{\nu}\nabla^{\nu}h\|_{L^2}+\sum_{\nu=0}^{k-1}\left\|(1+|x|)^{\nu+1}\nabla^{\nu}\left(\tilde f,\tilde g,\tilde r\right)\right\|_{L^2}\Bigg\},
\end{aligned} \tag{17}$$

where $C > 0$ is a constant depending only on μ, λ, κ.

3. The Proof of Theorem 1

In this section, we prove the main Theorem 1 for the existence of stationary solution of (6) by using the contraction mapping principle in $\tilde X_\varepsilon^{4,5,5,5}$. For this purpose, we study the following iterated equations

$$\begin{cases}
\operatorname{div} u+\dfrac{\tilde\rho_P}{\tilde\rho}(\tilde u\cdot\nabla)\varrho=h,\\
-\mu\Delta u-(\mu+\lambda)\nabla\operatorname{div} u+\nabla\varrho=-\bar\rho(\tilde u\cdot\nabla)\tilde u+\tilde f,\\
-\kappa\Delta\vartheta=-\bar\zeta_1(\tilde u\cdot\nabla)\tilde\vartheta+\tilde g,\\
-\Delta w=-(\tilde u\cdot\nabla)\tilde w+\tilde r,
\end{cases} \tag{18}$$

where $(\tilde\varrho,\tilde u,\tilde\vartheta,\tilde w)\in\tilde X_\varepsilon^{4,5,5,5}$ is given, $\tilde\rho_P=\rho_P(\bar P+\tilde\varrho,\bar\theta+\tilde\vartheta)$, $\bar\zeta_1=\zeta_1(\bar P,\bar\theta)$, etc., and

$$\begin{cases}
h=-\dfrac{\tilde\rho_\theta}{\tilde\rho}(\tilde u\cdot\nabla)\tilde\vartheta+\dfrac{H}{\tilde\rho},\ \tilde f=-(\tilde\rho-\bar\rho)(\tilde u\cdot\nabla)\tilde u-\nabla\tilde w\cdot\Delta\tilde w+\tilde\rho F,\\
\tilde g=-\left(\tilde\zeta_1-\bar\zeta_1\right)(\tilde u\cdot\nabla)\tilde\vartheta-\tilde\zeta_2(\tilde u\cdot\nabla)\tilde\varrho+\Psi(\tilde u)+\left|\Delta\tilde w+|\nabla\tilde w|^2\left(\tilde w+\bar d\right)\right|^2+\tilde\zeta_3H+G,\\
\tilde r=|\nabla\tilde w|^2\left(\tilde w+\bar d\right)+R,\ \tilde\zeta_1=\tilde\rho+\dfrac{\tilde\rho_\theta}{\tilde\rho}\left(\tilde\varrho+\bar P\right),\ \tilde\zeta_2=\dfrac{\tilde\rho_P}{\tilde\rho}\left(\tilde\varrho+\bar P\right),\ \tilde\zeta_3=\dfrac{\tilde\varrho+\bar P}{\tilde\rho}.
\end{cases} \tag{19}$$

3.1. Construction of Solution Map $\mathscr F$ for (18). Now, we devote ourselves to establishing a solution map for (18). To begin with, we derive the weighted L^2 estimates on the solution of (18) by applying Theorem 2. In fact, let

$$a=\frac{\tilde\rho_P}{\tilde\rho}\tilde u,\ b_1=c_1=\bar\rho^{1/2}\tilde u,\ b_2=\bar\zeta_1\tilde u,\ c_2=\tilde\vartheta,\ b_3=\tilde u,\ c_3=\tilde w \tag{20}$$

and $h, \tilde f, \tilde g, \tilde r$ in Theorem 2 be denoted as in (19). By the Sobolev inequality, we can choose $\varepsilon > 0$ sufficiently small such that $\bar\rho/2 \le \tilde\rho \le 2\bar\rho$ and η in (15) satisfy $\eta \le \eta_0$. It is straightforward to check that (15) holds for $k = 4$, and

$$\begin{aligned}
&\|(1+|x|)h\|_{L^2}+\sum_{\nu=1}^{4}\|(1+|x|)^{\nu}\nabla^{\nu}h\|_{L^2}\\
&\quad+\sum_{\nu=0}^{3}\left\|(1+|x|)^{\nu+1}\nabla^{\nu}\left(\tilde f,\tilde g,\tilde r\right)\right\|_{L^2}\le C\left(\varepsilon^2+K_0\right),
\end{aligned} \tag{21}$$

with K_0 defined by

$$K_0:=\sum_{\nu=0}^{3}\left\|(1+|x|)^{\nu+1}\nabla^{\nu}(H,F,G,R)\right\|_{L^2}+\left\|(1+|x|)^4\nabla^4H\right\|_{L^2}<\infty. \tag{22}$$

Thus, one has the desired result by applying Theorem 2 with $k = 4$ for (18).

Lemma 3. *Let (H, F, G, R) satisfy (22). Then, there exists a positive constants ε_0 such that if $\varepsilon \le \varepsilon_0$, the problem (18) with $(\tilde\varrho,\tilde u,\tilde\vartheta,\tilde w)\in\tilde X_\varepsilon^{4,5,5,5}$ has a solution $(\varrho,u,\vartheta,w)\in\widehat H^4\times\widehat H^5\times\widehat H^5\times\widehat H^5$ satisfying*

$$\begin{aligned}
&\|(\varrho,u,\vartheta,w)\|_{L^6}+\sum_{\nu=1}^{4}\|(1+|x|)^{\nu}\nabla^{\nu}\varrho\|_{L^2}\\
&\quad+\sum_{\nu=1}^{5}\left\|(1+|x|)^{\nu-1}(\nabla^{\nu}u,\nabla^{\nu}\vartheta,\nabla^{\nu}w)\right\|_{L^2}\le C\left(\varepsilon^2+K_0\right),
\end{aligned} \tag{23}$$

where $C > 0$ is a constant depending only on $\bar\rho, \bar\theta, \bar d, \lambda, \mu, \kappa$.

To continue, we cite a lemma in [22] which will be used to establish the L^∞ norm of the solution to (18).

Lemma 4 (see [22]). *Let $E(x)$ be a scalar function satisfying*

$$|\partial_x^\alpha E(x)|\le\frac{C_\alpha}{|x|^{|\alpha|+1}},\ (|\alpha|=0,1,2). \tag{24}$$

(i) If $\varphi(x)$ is a smooth scalar function in the form: $\varphi = \operatorname{div}\varphi_1 + \varphi_2$ and satisfies

$$L_1(\varphi)=\left\|(1+|x|)^3\varphi\right\|_{L^\infty}+\left\|(1+|x|)^2\varphi_1\right\|_{L^\infty}+\|\varphi_2\|_{L^1}<\infty, \tag{25}$$

then for any multi-index α with $|\alpha| = 0, 1$, one has

$$|\partial_x^\alpha (E * \varphi)(x)| \le \frac{C_\alpha}{|x|^{|\alpha|+1}} L_1(\varphi). \tag{26}$$

(ii) If $\varphi(x)$ is a smooth scalar function in the form: $\varphi = \varphi_1\varphi_2$ and satisfies

$$L_2(\varphi) = \left\|(1+|x|)^2\varphi\right\|_{L^\infty} + \left\|(1+|x|)^3(\nabla\varphi_1)\varphi_2\right\|_{L^\infty} + \left\|(1+|x|)^3\varphi_1(\nabla\varphi_2)\right\|_{H^1} < \infty, \tag{27}$$

then for any multi-index α with $|\alpha| = 1, 2$, one has

$$|\partial_x^\alpha (E * \varphi)(x)| \le \frac{C_\alpha}{|x|^{|\alpha|}} L_2(\varphi). \tag{28}$$

Here, the constant C_α depends only on α.

According to Lemma 3, we can introduce the solution map for (18): $\mathcal{F} : \tilde{X}_\varepsilon^{4,5,5,5} \longrightarrow \widehat{H}^4 \times \widehat{H}^5 \times \widehat{H}^5 \times \widehat{H}^5$ by $\mathcal{F}(\tilde{\varrho}, \tilde{u}, \tilde{\vartheta}, \tilde{w}) = (\varrho, u, \vartheta, w)$. In fact, in the following proposition, we can see that $\mathcal{F}$ maps $\tilde{X}_\varepsilon^{4,5,5,5}$ into itself. That is, $\mathcal{F}(\tilde{\varrho}, \tilde{u}, \tilde{\vartheta}, \tilde{w}) = (\varrho, u, \vartheta, w) \in \tilde{X}_\varepsilon^{4,5,5,5}$.

Proposition 5. *There exists $c_0 > 0$ such that for sufficiently small $\varepsilon > 0$, if (H, F, G, R) satisfies*

$$K := \|(H, F, G, R)\|_{\mathcal{S}} + \left\|(1+|x|)^4\nabla^4 H\right\|_{L^2} + \left\|(1+|x|)^{-1} H\right\|_{L^1} \le c_0\varepsilon, \tag{29}$$

then (18) with $(\tilde{\varrho}, \tilde{u}, \tilde{\vartheta}, \tilde{w}) \in \tilde{X}_\varepsilon^{4,5,5,5}$ has a solution $(\varrho, u, \vartheta, w) = \mathcal{F}(\tilde{\varrho}, \tilde{u}, \tilde{\vartheta}, \tilde{w}) \in \tilde{X}_\varepsilon^{4,5,5,5}$.

Proof. We prove this result by two steps. □

Step 1. We begin with the L^∞-norm of the solution $(\varrho, u, \vartheta, w)$ to (18). From the Helmholtz decomposition and Fourier transform, the solution of (18) can be written of the form and compare [22]:

$$u = v + \nabla p, \varrho = \psi + (2\mu + \lambda)\Delta p, \vartheta = E_0 * \Theta, w = E_0 * \phi, \tag{30}$$

where

$$\begin{cases} v_j(x) = \displaystyle\sum_{i=1}^{3} E_{ij} * f_i(x), \\ p = E_0 * \varphi(x), \\ \psi = E_0 * (\operatorname{div} f), \end{cases}$$
$$\begin{cases} E_{ij} = \dfrac{1}{8\pi\mu}\left(\dfrac{\delta_{ij}}{|x|} - \dfrac{x_i x_j}{|x|^3}\right), \ E_0 = -\dfrac{1}{4\pi|x|}, \\ f_i = -\tilde{\rho}(\tilde{u}\cdot\nabla)\tilde{u}_i + \tilde{f}_i, \\ \varphi = -\dfrac{\tilde{\rho}_P}{\tilde{\rho}}(\tilde{u}\cdot\nabla)\varrho + h, \\ \Theta = \dfrac{1}{\kappa}\left(\tilde{\zeta}_1(\tilde{u}\cdot\nabla)\tilde{\vartheta} - \tilde{g}\right), \\ \phi = (\tilde{u}\cdot\nabla)\tilde{w} - \tilde{r}. \end{cases} \tag{31}$$

Now, we apply Lemma 4 to get the estimate for $(\varrho, u, \vartheta, w)$. As a start, we study more details of f in order to estimate u and ϱ. Since $(\tilde{\varrho}, \tilde{u}, \tilde{\vartheta}, \tilde{w}) \in \tilde{X}_\varepsilon^{4,5,5,5}$, there exists $\tilde{U}_1 = (\tilde{U}_1^i)_{1\le i\le 3}$ and $\tilde{U}_2$ such that

$$\operatorname{div}\ \tilde{u} = \operatorname{div}\ \tilde{U}_1 + \tilde{U}_2, \text{and } \left\|(1+|x|)^3\tilde{U}_1\right\|_{L^\infty} + \left\|(1+|x|)^{-1}\tilde{U}_2\right\|_{L^1} \le \varepsilon. \tag{32}$$

This in turn yields

$$\begin{aligned} f_i &= -\tilde{\rho}(\tilde{u}\cdot\nabla)\tilde{u}_i - \nabla\tilde{w}\cdot\Delta\tilde{w} + \tilde{\rho}F_i \\ &= \operatorname{div}\left(-\tilde{\rho}\tilde{u}_i\tilde{u} + \tilde{\rho}\tilde{u}_i\tilde{U}_1 + \tilde{\rho}F_{1,i}\right) + \left(-\tilde{\rho}(\tilde{U}_1\cdot\nabla)\tilde{u}_i - \tilde{u}_i(\tilde{U}_1\cdot\nabla)\tilde{\rho}\right. \\ &\quad \left. + \tilde{\rho}\tilde{u}_i\tilde{U}_2 - \nabla\tilde{w}\cdot\Delta\tilde{w} - \nabla\tilde{\rho}\cdot F_{1,i} + \tilde{\rho}F_{2,i}\right) =: \operatorname{div}\ f_{1,i} + f_{2,i}. \end{aligned} \tag{33}$$

In view of (32), the Sobolev inequality and the fact that $(\tilde{\varrho}, \tilde{u}, \tilde{\vartheta}, \tilde{w}) \in \tilde{X}_\varepsilon^{4,5,5,5}$, a direct computation gives that

$$\left\|(1+|x|)^2 f_{1,i}\right\|_{L^\infty} + \left\|f_{2,i}\right\|_{L^1} + \left\|(1+|x|)^3 f_i\right\|_{L^\infty} \le C\left(\varepsilon^2 + K_1\right), \tag{34}$$

with K_1 defined by

$$K_1 := \left\|(1+|x|)^3 F\right\|_{L^\infty} + \left\|(1+|x|)^2 F_1\right\|_{L^\infty} + \|F_2\|_{L^1}. \tag{35}$$

By Lemma 4(i), we thus obtain

$$|x|\left|v_j\right|, |x|^2\left|\nabla v_j\right|, |x|^2|\psi| \le C\left(\varepsilon^2 + K_1\right). \tag{36}$$

Next, we calculate the estimate for p. Notice that

$$\varphi = -\sum_{i=1}^{3}\frac{\tilde{\rho}_P}{\tilde{\rho}}\tilde{u}_i\nabla_i\varrho + \left(-\sum_{i=1}^{3}\frac{\tilde{\rho}_\theta}{\tilde{\rho}}\tilde{u}_i\nabla_i\tilde{\vartheta} + \frac{H}{\tilde{\rho}}\right) =: -\sum_{i=1}^{3} q_1^i q_2^i + \chi. \tag{37}$$

Since $(\tilde{\varrho}, \tilde{u}, \tilde{\vartheta}, \tilde{w}) \in \tilde{X}_{\varepsilon}^{4,5,5,5}$, it follows from Lemma 3 that

$$\begin{aligned}&\left\|(1+|x|)^2 q_1^i q_2^i\right\|_{L^\infty} + \left\|(1+|x|)^3\left(\nabla q_1^i\right) q_2^i\right\|_{L^\infty} \\ &\quad + \left\|(1+|x|)^3 q_1^i\left(\nabla q_2^i\right)\right\|_{H^1} \\ &\quad \le C\left(\varepsilon^2 + K_0\right), \\ &\left\|(1+|x|)^2\chi\right\|_{L^\infty} + \left\|(1+|x|)^3\nabla\chi\right\|_{L^\infty} \\ &\quad \le C\left(\varepsilon^2 + K_0\right).\end{aligned} \tag{38}$$

Using Lemma 4(ii), we immediately find

$$|x||\nabla p|, |x|^2\left|\nabla^2 p\right| \le C\left(\varepsilon^2 + K_0\right). \tag{39}$$

Hence, according to (30), (36), and (39), we can conclude the estimate for u and ϱ as

$$|x||u|, |x|^2|\nabla u|, |x|^2|\varrho| \le C\left(\varepsilon^2 + K_0 + K_1\right). \tag{40}$$

To continue, we treat the estimate for ϑ. To this end, we first write Θ in the form

$$\begin{aligned}\Theta &= \frac{1}{\kappa}\left(\tilde{\zeta}_1(\tilde{u}\cdot\nabla)\tilde{\vartheta} + \tilde{\zeta}_2(\tilde{u}\cdot\nabla)\tilde{\varrho} - \Psi(\tilde{u}) - \left|\Delta\tilde{w} + |\nabla\tilde{w}|^2\left(\tilde{w}+\bar{d}\right)\right|^2 - \tilde{\zeta}_3 H - G\right) \\ &= \frac{1}{\kappa}\operatorname{div}\left(\left(\tilde{\zeta}_1\tilde{\vartheta} + \tilde{\zeta}_2\tilde{\varrho}\right)\left(\tilde{u} - \tilde{U}_1\right) - \tilde{\zeta}_3 H_1 - G_1\right) + \frac{1}{\kappa}\left(\left(\tilde{U}_1\cdot\nabla\right)\left(\tilde{\zeta}_1\tilde{\vartheta} + \tilde{\zeta}_2\tilde{\varrho}\right)\right. \\ &\quad - \left(\tilde{\zeta}_1\tilde{\vartheta} + \tilde{\zeta}_2\tilde{\varrho}\right)\tilde{U}_2 - \tilde{\vartheta}\tilde{u}\nabla\tilde{\zeta}_1 - \tilde{\varrho}\tilde{u}\nabla\tilde{\zeta}_2 - \Psi(\tilde{u}) - \left|\Delta\tilde{w} + |\nabla\tilde{w}|^2\left(\tilde{w}+\bar{d}\right)\right|^2 \\ &\quad \left. + \nabla\tilde{\zeta}_3\cdot H_1 - \tilde{\zeta}_3 H_2 - G_2\right) =: \operatorname{div}\,\Theta_1 + \Theta_2.\end{aligned} \tag{41}$$

This together with $(\tilde{\varrho}, \tilde{u}, \tilde{\vartheta}, \tilde{w}) \in \tilde{X}_{\varepsilon}^{4,5,5,5}$ and (32) that

$$\left\|(1+|x|)^2\Theta_1\right\|_{L^\infty} + \left\|\Theta_2\right\|_{L^1} + \left\|(1+|x|)^3\Theta\right\|_{L^\infty} \le C\left(\varepsilon^2 + K_2\right), \tag{42}$$

with K_2 defined by

$$K_2 := \left\|(1+|x|)^3(H,G)\right\|_{L^\infty} + \left\|(1+|x|)^2\left(H_1,G_1\right)\right\|_{L^\infty} + \left\|\left(H_2,G_2\right)\right\|_{L^1}. \tag{43}$$

By Lemma 4(i), it holds that

$$|x||\vartheta|, |x|^2|\nabla\vartheta| \le C\left(\varepsilon^2 + K_2\right). \tag{44}$$

As to the estimate for w, we note that ϕ can be rewritten as

$$\begin{aligned}\phi &= -(\tilde{u}\cdot\nabla)\tilde{w} + |\nabla\tilde{w}|^2\left(\tilde{w}+\bar{d}\right) + R \\ &= \operatorname{div}\left(-\tilde{u}\tilde{w} + \tilde{U}_1\tilde{w} + R_1\right) \\ &\quad + \left[\left(-\tilde{U}_1\cdot\nabla\right)\tilde{w} + \tilde{w}\tilde{U}_2 + |\nabla\tilde{w}|^2\left(\tilde{w}+\bar{d}\right) + R_2\right] \\ &=: \operatorname{div}\ \phi_1 + \phi_2.\end{aligned} \tag{45}$$

By the Sobolev inequality and the fact that $(\tilde{\varrho}, \tilde{u}, \tilde{\vartheta}, \tilde{w}) \in \tilde{X}_{\varepsilon}^{4,5,5,5}$, we obtain

$$\left\|(1+|x|)^2\phi_1\right\|_{L^\infty} + \left\|\phi_2\right\|_{L^1} + \left\|(1+|x|)^3\phi\right\|_{L^\infty} \le C\left(\varepsilon^2 + K_3\right), \tag{46}$$

with K_3 defined by

$$K_3 := \left\|(1+|x|)^3 R\right\|_{L^\infty} + \left\|(1+|x|)^2 R_1\right\|_{L^\infty} + \left\|R_2\right\|_{L^1}. \tag{47}$$

With this help and Lemma 4(i), one has

$$|x||w|, |x|^2|\nabla w| \le C\left(\varepsilon^2 + K_3\right). \tag{48}$$

Next, for the case of $|x| < 1$, in terms of Lemma 3 and the Sobolev inequality, we immediately deduce that

$$\left\|\nabla^\nu(\varrho, u, \vartheta, w)\right\|_{L^\infty} \le C\left\|\nabla^{\nu+1}(\varrho, u, \vartheta, w)\right\|_{H^1} \le \left(\varepsilon^2 + K_0\right), \nu = 0, 1. \tag{49}$$

Consequently, combining (40), (44), (48), and (49), one has

$$\left\|(1+|x|)^2\varrho\right\|_{L^\infty} + \sum_{\nu=0}^{1}\left\|(1+|x|)^{\nu+1}\nabla^\nu(u,\vartheta,w)\right\|_{L^\infty} \le C\left(\varepsilon^2 + K\right). \tag{50}$$

Step 2. By Lemma 3 and (50), it follows that (18) admits a solution $(\varrho, u, \vartheta, w)$ satisfying

$$\left\|(\varrho, u, \vartheta, w)\right\|_{X^{4,5,5,5}} \le \bar{C}\left(\varepsilon^2 + K\right) \le C\varepsilon, \tag{51}$$

where the constants $\bar{C}, C > 0$ depend only on $\bar{\rho}, \bar{\theta}, \bar{d}, \mu, \lambda, \kappa$. To complete our proof, we need to study more details on u. To this end, let us define U_1 and U_2 as

$$U_1 = -\frac{\tilde{\rho}_P}{\tilde{\rho}}\tilde{u}\varrho,\ U_2 = \varrho\ \operatorname{div}\left(\frac{\tilde{\rho}_P}{\tilde{\rho}}\tilde{u}\right) - \frac{\tilde{\rho}_\theta}{\tilde{\rho}}(\tilde{u}\cdot\nabla)\tilde{\vartheta} + \frac{H}{\tilde{\rho}}. \tag{52}$$

Thus, from $(18)_1$, the solution u can be written as

$$\operatorname{div}\ u = \operatorname{div}\ U_1 + U_2. \tag{53}$$

In addition, by $(\tilde{\varrho}, \tilde{u}, \tilde{\vartheta}, \tilde{w}) \in \tilde{X}_{\varepsilon}^{4,5,5,5}$ and (50), we can get

$$\begin{aligned}&\left\|(1+|x|)^3 U_1\right\|_{L^\infty} + \left\|(1+|x|)^{-1} U_2\right\|_{L^1} \\ &\quad \le C\left(\varepsilon^2 + K + \left\|(1+|x|)^{-1} H\right\|_{L^1}\right) \le C\varepsilon.\end{aligned} \tag{54}$$

The proof of this proposition is completed.

3.2. Contraction of the Solution Map $\mathcal{F}$. Now, we are in a position to prove the solution map $\mathcal{F}$ for (18) is contractive. Assume that $(\tilde{\varrho}^j, \tilde{u}^j, \tilde{\vartheta}^j, \tilde{w}^j) \in \tilde{X}_{\varepsilon}^{4,5,5,5}$ and $(\varrho^j, u^j, \vartheta^j, w^j) = \mathcal{F}(\tilde{\varrho}^j, \tilde{u}^j, \tilde{\vartheta}^j, \tilde{w}^j)$ for $j = 1, 2$. By (18), a straightforward calculation yields that

$$\begin{cases} \operatorname{div}\left(u^1-u^2\right)+\dfrac{\tilde{\rho}_P^1}{\tilde{\rho}^1}\left(\tilde{u}^1\cdot\nabla\right)\left(\varrho^1-\varrho^2\right)=h, \\ -\mu\Delta\left(u^1-u^2\right)-(\mu+\lambda)\nabla\operatorname{div}\left(u^1-u^2\right)+\nabla\left(\varrho^1-\varrho^2\right)=-\tilde{\rho}^2\left(\left(\tilde{u}^1-\tilde{u}^2\right)\cdot\nabla\right)\tilde{u}^1-\tilde{\rho}^2\left(\tilde{u}^2\cdot\nabla\right)\left(\tilde{u}^1-\tilde{u}^2\right)+\tilde{f}, \\ -\kappa\Delta\left(\vartheta^1-\vartheta^2\right)=-\tilde{\zeta}_1^2\left(\left(\tilde{u}^1-\tilde{u}^2\right)\cdot\nabla\right)\tilde{\vartheta}^1-\tilde{\zeta}_1^2\left(\tilde{u}^2\cdot\nabla\right)\left(\tilde{\vartheta}^1-\tilde{\vartheta}^2\right)+\tilde{g}, \\ -\Delta\left(w^1-w^2\right)=-\left(\left(\tilde{u}^1-\tilde{u}^2\right)\cdot\nabla\right)\tilde{w}^1-\left(\tilde{u}^2\cdot\nabla\right)\left(\tilde{w}^1-\tilde{w}^2\right)+\tilde{r}, \end{cases} \tag{55}$$

where

$$\begin{cases} h=-\left(\dfrac{\tilde{\rho}_P^1}{\tilde{\rho}^1}\tilde{u}^1-\dfrac{\tilde{\rho}_P^2}{\tilde{\rho}^2}\tilde{u}^2\right)\cdot\nabla\varrho^2-\left(\dfrac{\tilde{\rho}_\theta^1}{\tilde{\rho}^1}\left(\tilde{u}^1\cdot\nabla\right)\tilde{\vartheta}^1-\dfrac{\tilde{\rho}_\theta^2}{\tilde{\rho}^2}\left(\tilde{u}^2\cdot\nabla\right)\tilde{\vartheta}^2\right)+\left(\dfrac{1}{\tilde{\rho}^1}-\dfrac{1}{\tilde{\rho}^2}\right)H, \\ \tilde{f}=-\left(\tilde{\rho}^1-\tilde{\rho}^2\right)\left(\tilde{u}^1\cdot\nabla\right)\tilde{u}^1-\left(\nabla\tilde{w}^1-\nabla\tilde{w}^2\right)\cdot\Delta\tilde{w}^1-\nabla\tilde{w}^2\cdot\left(\Delta\tilde{w}^1-\Delta\tilde{w}^2\right)+\left(\tilde{\rho}^1-\tilde{\rho}^2\right)F, \\ \tilde{g}=-\left(\tilde{\zeta}_1^1-\tilde{\zeta}_1^2\right)\left(\tilde{u}^1\cdot\nabla\right)\tilde{\vartheta}^1-\left(\tilde{\zeta}_2^1-\tilde{\zeta}_2^2\right)\left(\tilde{u}^1\cdot\nabla\right)\tilde{\varrho}^1-\tilde{\zeta}_2^2\left(\left(\tilde{u}^1\cdot\nabla\right)\tilde{\varrho}^1-\left(\tilde{u}^2\cdot\nabla\right)\tilde{\varrho}^2\right)+\Psi\left(\tilde{u}^1\right)-\Psi\left(\tilde{u}^2\right)+\left|\Delta\tilde{w}^1+\left|\nabla\tilde{w}^1\right|^2\left(\tilde{w}^1+\bar{d}\right)\right|^2-\left|\Delta\tilde{w}^2+\left|\nabla\tilde{w}^2\right|^2\left(\tilde{w}^2+\bar{d}\right)\right|^2+\left(\tilde{\zeta}_3^1-\tilde{\zeta}_3^2\right)H, \\ \tilde{r}=\left(\left|\nabla\tilde{w}^1\right|^2-\left|\nabla\tilde{w}^2\right|^2\right)\left(\tilde{w}^1+\bar{d}\right)+\left|\nabla\tilde{w}^2\right|^2\left(\tilde{w}^1-\tilde{w}^2\right), \\ \tilde{\zeta}_1^j=\tilde{\rho}^j+\dfrac{\tilde{\rho}_\theta^j}{\tilde{\rho}^j}\left(\tilde{\varrho}^j+\bar{P}\right),\quad \tilde{\zeta}_2^j=\dfrac{\tilde{\rho}_P^j}{\tilde{\rho}^j}\left(\tilde{\varrho}^j+\bar{P}\right),\quad \tilde{\zeta}_3^j=\dfrac{\tilde{\varrho}^j+\bar{P}}{\tilde{\rho}^j},\quad j=1,2. \end{cases} \tag{56}$$

By the Sobolev inequality, it holds that

$$\begin{aligned} &\|(1+|x|)h\|_{L^2}+\sum_{\nu=1}^{3}\|(1+|x|)^\nu\nabla^\nu h\|_{L^2} \\ &\quad+\sum_{\nu=0}^{2}\left\|(1+|x|)^{\nu+1}\nabla^\nu\left(\tilde{f},\tilde{g},\tilde{r}\right)\right\|_{L^2} \\ &\leq C(\varepsilon+K_0)\left\|\left(\tilde{\varrho}^1-\tilde{\varrho}^2,\tilde{u}^1-\tilde{u}^2,\tilde{\vartheta}^1-\tilde{\vartheta}^2,\tilde{w}^1-\tilde{w}^2\right)\right\|_{X^{3,4,4,4}}, \end{aligned} \tag{57}$$

with K_0 defined in (22). By applying Theorem 2 with $k=3$ to (55), we obtain

$$\begin{aligned} &\left\|\left(\varrho^1-\varrho^2,u^1-u^2,\vartheta^1-\vartheta^2,w^1-w^2\right)\right\|_{L^6} \\ &\quad+\sum_{\nu=1}^{3}\left\|(1+|x|)^\nu\nabla^\nu\left(\varrho^1-\varrho^2\right)\right\|_{L^2} \\ &\quad+\sum_{\nu=1}^{4}\left\|(1+|x|)^{\nu-1}\left(\nabla^\nu\left(u^1-u^2\right),\nabla^\nu\left(\vartheta^1-\vartheta^2\right),\nabla^\nu\left(w^1-w^2\right)\right)\right\|_{L^2} \\ &\leq C(\varepsilon+K_0)\left\|\left(\tilde{\varrho}^1-\tilde{\varrho}^2,\tilde{u}^1-\tilde{u}^2,\tilde{\vartheta}^1-\tilde{\vartheta}^2,\tilde{w}^1-\tilde{w}^2\right)\right\|_{X^{3,4,4,4}}. \end{aligned} \tag{58}$$

With the same computations as used in (50), one can arrive at

$$\begin{aligned} &\left\|(1+|x|)^2\left(\varrho^1-\varrho^2\right)\right\|_{L^\infty}+\sum_{\nu=0}^{1}\left\|(1+|x|)^{\nu+1}\left(\nabla^\nu\left(u^1-u^2\right),\nabla^\nu\left(\vartheta^1-\vartheta^2\right),\nabla^\nu\left(w^1-w^2\right)\right)\right\|_{L^\infty} \\ &\leq C(\varepsilon+K_0)\left\|\left(\tilde{\varrho}^1-\tilde{\varrho}^2,\tilde{u}^1-\tilde{u}^2,\tilde{\vartheta}^1-\tilde{\vartheta}^2,\tilde{w}^1-\tilde{w}^2\right)\right\|_{X^{3,4,4,4}} \\ &\quad+C\varepsilon\left(\left\|(1+|x|)^3\left(\tilde{U}_1^1-\tilde{U}_1^2\right)\right\|_{L^\infty}+\left\|(1+|x|)^{-1}\left(\tilde{U}_2^1-\tilde{U}_2^2\right)\right\|_{L^1}\right), \end{aligned} \tag{59}$$

where $\tilde{U}_1^j,\tilde{U}_2^j(j=1,2)$ are functions satisfying

$$\operatorname{div}\ \tilde{u}^j=\operatorname{div}\ \tilde{U}_1^j+\tilde{U}_2^j,\quad \left\|(1+|x|)^3\tilde{U}_1^j\right\|_{L^\infty}+\left\|(1+|x|)^{-1}\tilde{U}_2^j\right\|_{L^1}\leq\varepsilon. \tag{60}$$

In addition, we denote U_1^j and $U_2^j(j=1,2)$ as

$$U_1^j=-\frac{\tilde{\rho}_P^j}{\tilde{\rho}^j}\tilde{u}^j\varrho^j,\quad U_2^j=\varrho^j\operatorname{div}\left(\frac{\tilde{\rho}_P^j}{\tilde{\rho}^j}\tilde{u}^j\right)-\frac{\tilde{\rho}_\theta^j}{\tilde{\rho}^j}\left(\tilde{u}^j\cdot\nabla\right)\tilde{\vartheta}^j+\frac{H}{\tilde{\rho}^j}. \tag{61}$$

We can derived from $(\tilde{\varrho}^j, \tilde{u}^j, \tilde{\vartheta}^j, \tilde{w}^j) \in \tilde{X}_\varepsilon^{4,5,5,5}$ that

$$\begin{aligned}
&\left\|(1+|x|)^3\left(U_1^1-U_1^2\right)\right\|_{L^\infty}+\left\|(1+|x|)^{-1}\left(U_2^1-U_2^2\right)\right\|_{L^1} \\
&\quad \le C\left(\varepsilon+\left\|(1+|x|)^{-1}H\right\|_{L^1}\right)\left\|\left(\tilde{\varrho}^1-\tilde{\varrho}^2, \tilde{u}^1-\tilde{u}^2, \tilde{\vartheta}^1-\tilde{\vartheta}^2, \tilde{w}^1-\tilde{w}^2\right)\right\|_{X^{3,4,4,4}},
\end{aligned} \tag{62}$$

which combined with (58) and (59) yields

$$\begin{aligned}
&\left\|\left(\varrho^1-\varrho^2, u^1-u^2, \vartheta^1-\vartheta^2, w^1-w^2\right)\right\|_{X^{3,4,4,4}} \\
&\quad +\left\|(1+|x|)^3\left(U_1^1-U_1^2\right)\right\|_{L^\infty}+\left\|(1+|x|)^{-1}\left(U_2^1-U_2^2\right)\right\|_{L^1} \\
&\quad \le C(\varepsilon+K))\left\|\left(\tilde{\varrho}^1-\tilde{\varrho}^2, \tilde{u}^1-\tilde{u}^2, \tilde{\vartheta}^1-\tilde{\vartheta}^2, \tilde{w}^1-\tilde{w}^2\right)\right\|_{X^{3,4,4,4}} \\
&\quad + C\varepsilon\left(\left\|(1+|x|)^3\left(\tilde{U}_1^1-\tilde{U}_1^2\right)\right\|_{L^\infty}+\left\|(1+|x|)^{-1}\left(\tilde{U}_2^1-\tilde{U}_2^2\right)\right\|_{L^1}\right).
\end{aligned} \tag{63}$$

With the above preparation in hand, we then have the following proposition.

Proposition 6. *Assume that (H, F, G, R) satisfies the estimate (for K defined in Proposition 5):*

$$K \le c_0\varepsilon, \tag{64}$$

for some $c_0>0$ and sufficiently small $\varepsilon>0$. Then, for $(\tilde{\varrho}^j, \tilde{u}^j, \tilde{\vartheta}^j, \tilde{w}^j) \in \tilde{X}_\varepsilon^{4,5,5,5}$ and $(\varrho^j, u^j, \vartheta^j, w^j)=\mathscr{F}(\tilde{\varrho}^j, \tilde{u}^j, \tilde{\vartheta}^j, \tilde{w}^j)$ $(j=1,2)$, one has

$$\begin{aligned}
&\left\|\left(\varrho^1-\varrho^2, u^1-u^2, \vartheta^1-\vartheta^2, w^1-w^2\right)\right\|_{X^{3,4,4,4}} \\
&\quad +\left\|(1+|x|)^3\left(U_1^1-U_1^2\right)\right\|_{L^\infty}+\left\|(1+|x|)^{-1}\left(U_2^1-U_2^2\right)\right\|_{L^1} \\
&\quad \le \frac{1}{2}\left(\left\|\left(\tilde{\varrho}^1-\tilde{\varrho}^2, \tilde{u}^1-\tilde{u}^2, \tilde{\vartheta}^1-\tilde{\vartheta}^2, \tilde{w}^1-\tilde{w}^2\right)\right\|_{X^{3,4,4,4}}\right. \\
&\quad \left.+\left\|(1+|x|)^3\left(\tilde{U}_1^1-\tilde{U}_1^2\right)\right\|_{L^\infty}+\left\|(1+|x|)^{-1}\left(\tilde{U}_2^1-\tilde{U}_2^2\right)\right\|_{L^1}\right),
\end{aligned} \tag{65}$$

where $(\tilde{U}_1^j, \tilde{U}_2^j)(j=1,2)$ satisfies (60), and $(U_1^j, U_2^j)(j=1,2)$ are defined by (61).

Therefore, by Proposition 5, Proposition 6, and the contraction mapping principle, we immediately get the existence and uniqueness of solution to system (6). The proof of Theorem 1 is now completed.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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