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Golden Section and Evolving Systems

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Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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ABSTRACT

We present a body of results that demonstrate that the golden section can be implemented in the design of evolving systems. Albeit we only develop the mathematics, both mathematician and computer expert should find the concept clear. Perhaps of particular interest to the number theorist is the observation that the sequence $T_n = 1,4,7,10,13,\dots$ defined by $t_{n+1} = t_n + 3, n \geq 1, t_1 = 1$, known as the Teleois number system, crops up in our results. Having shown in previous works how this sequence is closely related to the golden section, this manuscript gives further confirmation and the fact that Teleois numbers penetrate the golden section renders it a proportion of great splendour. Our results can find a wide range of applications from information technology to manufacturing.

Keywords: Cassini identity; evolving systems; Fibonacci sequence; golden section; Teleois numbers; transformation vector; zero transformation.

1. INTRODUCTION

Let an integer x satisfy

$$\begin{cases} y = round(x\varphi), \\ y - x = z, \\ x \neq round(z\varphi), \\ \varphi = \frac{1+\sqrt{5}}{2} \end{cases} \qquad (1.1)$$

x is called a parent number and is a seed value of a quasigeometric sequence H_n satisfying the relation

$$h_{n+1} = round(\varphi h_n), n \geq 1 \qquad (1.2)$$

studied in [1]. As a natural consequence of relation (1.2), H_n also satisfies

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$$h_{n+2} = h_{n+1} + h_n, n \geq 1 \quad (1.3)$$

but the converse is not true.

The work at hand is devoted to mathematically demonstrating that the golden section, denoted φ in equation (1.1) and formally defined by the ancient Greek mathematician Euclid in his seminal work *The Elements* [2] as the division of a line segment into extreme and mean ratio, can be implemented in (the design of) evolving systems. The concept makes much reliance upon the Cassini identity for H_n given generally as

$$\left.\begin{matrix} h_n h_{n+2} - {h_{n+1}}^2 = c(-1)^a, n \geq 1 \\ a = n \; or \; n+1 \end{matrix}\right\} \quad (1.4)$$

Countless scholars have worked on the application of the Cassini identity and Fibonacci numbers in computing science, especially cryptography, see [3-12].

We herein introduce transformations based on the Cassini rule and we obtain useful results. For analysis purposes, the sequence H_n shall be represented in the form

$$h_i = g_{n+i-1} \pm f_i, i \geq 1, n \geq 4 \quad (1.5a)$$

where

$$G_n: g_{n+1} = round(\varphi g_n), n \geq 1 \quad (1.5b)$$

and

$$F_n = 1,2,3,5,8, \dots \quad (1.6)$$

The concept of parent number as defined above has enabled us to regenerate not only the sequence (1.6) but also the "Lucas numbers" through the sequence

$$H_n = 7,11,18,29,47, \dots \quad (1.7)$$

We would like to assemble a sequence L_n defined by

$$l_n = f_i + f_{i+2}, n \geq 1 \quad (1.8)$$

thus

$$L_n = 4,7,11,18,29, \dots \quad (1.9)$$

Now let

$$2l_n = l_n^{'}, n \geq 1 \quad (1.10)$$

It follows that

$$L_n^{'} = 8,14,22,36,58, \dots \quad (1.11)$$

The sequence (1.9) has profound significance to the results of this work. The sequence (1.11), call it the "double Lucas numbers", is important in the study of symmetry. One may find the result of Theorem 1.1 interesting.

Theorem 1.1

Consider two sequences P_n and Q_n such that

$$\left.\begin{matrix} p_i = h_{n+i-1} + f_i \\ q_i = h_{n+i-1} - f_i \end{matrix}\right\} i \geq 1, n \geq 5 \quad (1.12)$$

It holds that

$$(p_i + p_{i+2}) - (q_i + q_{i+2}) = l_i^{'} \quad (1.13)$$

Proof

$$\begin{aligned} p_i + p_{i+2} &= (h_{n+i-1} + f_i) + (h_{n+i+1} + f_{i+2}) \\ &= h_{n+i-1} + f_i + h_{n+i+1} + f_{i+2} \end{aligned} \quad (1.14)$$

$$\begin{aligned} q_i + q_{i+2} &= (h_{n+i-1} - f_i) + (h_{n+i+1} - f_{i+2}) \\ &= h_{n+i-1} - f_i + h_{n+i+1} - f_{i+2} \end{aligned} \quad (1.15)$$

$$\begin{aligned} (1.14) - (1.15) &= h_{n+i-1} + f_i + h_{n+i+1} + f_{i+2} - h_{n+i-1} + f_i - h_{n+i+1} + f_{i+2} \\ &= 2(f_i + f_{i+2}) \\ &= 2l_i \\ &= {l_i}^{'} \end{aligned} \quad (1.16)$$

2. RESULTS

In our results, the designations H_n, G_n, F_n, L_n refer to the sequences (1.2), (1.5b), (1.6), and (1.9) respectively.

2.1 Zero Transformation

Let the sequence H_n be defined by $h_i = g_{n+i-1} - f_i, i \geq 1, n \geq 4$. The Cassini identity is given by

$$h_i h_{i+2} - {h_{i+1}}^2 = c(-1)^{i+1}, i \geq 1 \quad (2.1)$$

see Theorem 2.3.

Let the sequence H_n be defined by $h_i = g_{n+i-1} + f_i, i \geq 1, n \geq 4$. The Cassini identity is given by

$$h_i h_{i+2} - {h_{i+1}}^2 = c(-1)^i, i \geq 1 \quad (2.2)$$

see Theorem 2.4.

Let the constant *c* be the Cassini value of H_n. Take $h_i, i \geq 5$. Let

$$\left.\begin{aligned} h_i - 1 = p_1 \\ h_i + 1 = q_1 \end{aligned}\right\} \tag{2.3}$$

where $\mathrm{P_n}$ and $\mathrm{Q_n}$ also satisfy the relation (1.2). Now take p_j and $q_j, j \geq 4$. Consider the sequences

$$\left.\begin{aligned} p_j - 1, p_{j+1} - 2, p_{j+2} - 3, p_{j+3} - 5, \ldots : c = c_1 \\ p_j + 1, p_{j+1} + 2, p_{j+2} + 3, p_{j+3} + 5, \ldots : c = c_2 \\ q_j - 1, q_{j+1} - 2, q_{j+2} - 3, q_{j+3} - 5, \ldots : c = c_3 \\ q_j + 1, q_{j+1} + 2, q_{j+2} + 3, q_{j+3} + 5, \ldots : c = c_4 \end{aligned}\right\} \tag{2.4}$$

Let

$$(h_i)_j = (c_1, c_2, c_3, c_4) \tag{2.5}$$

and

$$z = c_1 + c_2 + c_3 + c_4 \tag{2.6}$$

Theorem 2.1

For the sequence H_n, let equations (2.3) to (2.6) hold. $z = 4(h_{i+j-4} + h_{i+j-2})$.

Proof

$$c_1 + c_2 = 6f_j + 2f_{j+2} + 8h_{i+j} - 8f_{j+1} - 6h_{i+j-1} - 2h_{i+j+1} \tag{2.7}$$

$$c_3 + c_4 = 8h_{i+j} + 8f_{j+1} - 6f_j - 2f_{j+2} - 6h_{i+j-1} - 2h_{i+j+1} \tag{2.8}$$

$$\begin{aligned} z &= 16h_{i+j} - 12h_{i+j-1} - 4h_{i+j+1} \\ &= 4(4h_{i+j} - 3h_{i+j-1} - h_{i+j+1}) \\ &= 4(4h_{i+j} - 3h_{i+j-1} - h_{i+j} - h_{i+j-1}) \\ &= 4(3h_{i+j} - 4h_{i+j-1}) \\ &= 4(h_{i+j-4} + h_{i+j-2}) \end{aligned} \tag{2.9}$$

Proof is complete.

Now let $(h_j)_i = (c_1', c_2', c_3', c_4')$. Let $j = i + 1$. For even i, let the transformation vector for the mapping

$$(h_i)_j \to (h_j)_i, i \geq 4 \tag{2.10}$$

be given by

$$v = (c_1 - c_4', c_2 - c_1', c_3 - c_2', c_4 - c_3') \tag{2.11}$$

For odd i, let the transformation vector for the mapping

$$(h_i)_j \to (h_j)_i, i \geq 5 \tag{2.12}$$

be given by

$$v = (c_1 - c_2', c_2 - c_3', c_3 - c_4', c_4 - c_1') \tag{2.13}$$

Theorem 2.2: zero transformation

Consider the sequence H_n. For the mapping $(h_n)_{n+1} \to (h_{n+1})_n, n \geq 4$, let the transformation vector be given by $v = (w_1, w_2, w_3, w_4)$. $\sum_{k=1}^{4} w_k = 0$.

Proof

$$(h_n)_{n+1} = (c_1, c_2, c_3, c_4)$$

From Theorem 2.1,

$$z = 4\left(h_{n+(n+1)-4} + h_{n+(n+1)-2}\right) = 4(h_{2n-3} + h_{2n-1}) \tag{2.14}$$

$$(h_{n+1})_n = (c_1', c_2', c_3', c_4')$$

Again from Theorem 2.1,

$$z' = 4\left(h_{(n+1)+n-4} + h_{(n+1)+n-2}\right) = 4(h_{2n-3} + h_{2n-1}) \tag{2.15}$$

Notice that Equations (2.14) and (2.15) are equal, therefore transformation vector $v = 0$.

2.2 Direct Computation of Transformation Vector

We first need to state Theorems 2.3 and 2.4.

Theorem 2.3

Let the sequence H_n be defined by $h_i = g_{n+i-1} - f_i, i \geq 1, n \geq 4$. *The Cassini identity is given by*

$h_i h_{i+2} - {h_{i+1}}^2 = c(-1)^{i+1}, i \geq 1$, *where*

$$\left.\begin{aligned} c = g_{n-1} + g_{n-3} + c_g - 1 \\ c_g = g_n g_{n+2} - g_{n+1}^2 \end{aligned}\right\} \tag{2.16}$$

Proof

We provide proof by the Principle of Mathematical Induction. Base case: $i = 1$,

$$\begin{aligned} h_1 h_3 - h_2^2 &= (g_n - 1)(g_{n+2} - 3) - (g_{n+1} - 2)^2 \\ &= g_n g_{n+2} - 3g_n - g_{n+2} - g_{n+1}^2 + 4g_{n+1} - 1 \\ &= g_{n-1} + g_{n-3} + c_g - 1 \\ &= c(-1)^2 \end{aligned}$$

Inductive Hypothesis: Since Identity is true for $i = 1$, *assume it is also true for* $i = k \geq 1$, *that is,*
$h_k h_{k+2} - {h_{k+1}}^2 = c(-1)^{k+1}, k \geq 1$

Inductive Conclusion: Truth must be established for $i = k + 1$, *that is*

$$h_{k+1}h_{k+3} - {h_{k+2}}^2 = c(-1)^{k+2}, k \geq 1$$

We have that

$$c(-1)^{k+2} = -c(-1)^{k+1} = h_{k+1}^2 - h_k h_{k+2}$$

To complete the proof we need to show that

$$h_{k+1}h_{k+3} - {h_{k+2}}^2 = h_{k+1}^2 - h_k h_{k+2}$$

Notice that

$$\begin{aligned} &h_{k+1}h_{k+3} - {h_{k+2}}^2 \\ &= h_{k+1}(h_{k+1} + h_{k+2}) - h_{k+2}^2 \\ &= h_{k+1}^2 - h_{k+2}(h_{k+2} - h_{k+1}) \\ &= h_{k+1}^2 - h_k h_{k+2} \end{aligned}$$

Proof is complete.

Theorem 2.4

Let a the sequence H_n *be defined by* $h_i = g_{n+i-1} + f_i, i \geq 1, n \geq 4.$ *The Cassini identity is given by*

$h_i h_{i+2} - {h_{i+1}}^2 = c(-1)^i, i \geq 1$, *where*

$$\left.\begin{aligned} c &= g_{n-1} + g_{n-3} - c_g + 1 \\ c_g &= g_n g_{n+2} - g_{n+1}^2 \end{aligned}\right\} \quad (2.17)$$

Proof

Follow proof to Theorem 2.3.

Theorems 2.3 and 2.4 are crucial to Theorems 2.5 to 2.12 that deal with the direct computation of transformation vector.

Theorem 2.5

Let the sequence H_n *be such that*

$h_i = g_{n+i-1} - f_i, i \geq 1,\ g_n = q_{m+n-1} + f_n, m, n \geq 4, n \text{ is even}.$ *Consider the transformation* $(h_i)_{i+1} \rightarrow (h_{i+1})_i$ (2.18)

Let the transformation vector be given by

$v = (w_1, w_2, w_3, w_4)$. *When* i *is even,*

$$\left.\begin{aligned} w_1 &= h_{i+1} + h_{i-1} - l_{i-1} - 2(g_{n-1} + g_{n-3} + c_g) \\ w_2 &= h_{i-2} + h_{i-4} - l_{i-4} + 2(g_{n-1} + g_{n-3} + c_g) \\ w_3 &= l_{i-1} - h_{i+1} - h_{i-1} - 2(g_{n-1} + g_{n-3} + c_g) \\ w_4 &= l_{i-4} - h_{i-2} - h_{i-4} + 2(g_{n-1} + g_{n-3} + c_g) \\ c_g &= g_n g_{n+2} - g_{n+1}^2 \end{aligned}\right\} \quad (2.19)$$

when i *is odd,*

$$\left.\begin{aligned} w_1 &= h_{i-2} + h_{i-4} - l_{i-4} - 2(g_{n-1} + g_{n-3} + c_g) \\ w_2 &= h_{i+1} + h_{i-1} - l_{i-1} + 2(g_{n-1} + g_{n-3} + c_g) \\ w_3 &= l_{i-4} - h_{i-2} - h_{i-4} - 2(g_{n-1} + g_{n-3} + c_g) \\ w_4 &= l_{i-1} - h_{i+1} - h_{i-1} + 2(g_{n-1} + g_{n-3} + c_g) \\ c_g &= g_n g_{n+2} - g_{n+1}^2 \end{aligned}\right\} \quad (2.20)$$

Proof

<u>Scenario I:</u> i *is even*
Given $(h_i)_{i+1} \rightarrow (h_{i+1})_i$, *let*

$$(h_i)_{i+1} = (c_1, c_2, c_3, c_4)$$
$$(h_{i+1})_i = (c_1', c_2', c_3', c_4')$$

Since i *is even, equation (2.11) gives the transformation vector as* $v = (c_1 - c_4', c_2 - c_1', c_3 - c_2', c_4 - c_3')$. *Consider* $(h_i)_{i+1}$. *Using equation (2.5),* $j = i + 1$. *It follows* $p_1 = h_i - 1$. *This means*

$$\left.\begin{aligned} p_{i+1} &= h_{2i} - f_{i+1} \\ p_{i+2} &= h_{2i+1} - f_{i+2} \\ p_{i+3} &= h_{2i+2} - f_{i+3} \end{aligned}\right\} \quad (2.21)$$

$$\begin{aligned} &(p_{i+1} - 1)(p_{i+3} - 3) - (p_{i+2} - 2)^2 \\ &= (h_{2i} - f_{i+1} - 1)(h_{2i+2} - f_{i+3} - 3) - (h_{2i+1} - f_{i+2} - 2)^2 \\ &= h_{2i}h_{2i+2} - f_{i+3}h_{2i} - 3h_{2i} - f_{i+1}h_{2i+2} + f_{i+1}f_{i+3} \\ &\quad + 3f_{i+1} - h_{2i+1}^2 + 2f_{i+2}h_{2i+1} \\ &\quad + 4h_{2i+1} - 4f_{i+2} \\ &\quad - f_{i+2}^2 - 4 - h_{2i+2} + f_{i+3} + 3 \end{aligned} \quad (2.22)$$

With even i, *from Theorem 2.3,*

$$h_{2i}h_{2i+2} - h_{2i+1}^2 = -(g_{n-1} + g_{n-3} + c_g - 1)$$

Also notice that
$f_{i+1}f_{i+3} - f_{i+2}^2 = -1$, *therefore Equation (2.22) reduces to*

$$\begin{aligned} &2f_{i+2}h_{2i+1} + 4h_{2i+1} - f_{i+3}h_{2i} - 3h_{2i} - f_{i+1}h_{2i+2} \\ &\quad - h_{2i+2} - (3f_i + f_{i+2} - f_{i+3}) \\ &\quad -(g_{n-1} + g_{n-3} + c_g) - 1 \end{aligned} \quad (2.23)$$

Since i *is even, it follows* c_1 *is given by Equation (2.23). Now consider* $(h_{i+1})_i$.

$q_1^{'} = h_{i+1} + 1$
$q_i^{'} = h_{2i} + f_i, q_{i+1}^{'} = h_{2i+1} + f_{i+1}, q_{i+2}^{'} = h_{2i+2} + f_{i+2}$

$$(q_i^{'} + 1)(q_{i+2}^{'} + 3) - (q_{i+1}^{'} + 2)^2 = (h_{2i} + f_i + 1)(h_{2i+2} + f_{i+2} + 3) - (h_{2i+1} + f_{i+1} + 2)^2 \quad (2.24)$$

$$= h_{2i}h_{2i+2} + f_{i+2}h_{2i} + 3h_{2i} + f_i h_{2i+2} + f_i f_{i+2} + 3f_i + h_{2i+2} + f_{i+2} + 3 - h_{2i+1}^2 - 2f_{i+1}h_{2i+1} - 4h_{2i+1} - 4f_{i+1} - f_{i+1}^2 - 4$$

Since i is even, it follows $c_4^{'}$ is given by the negation of Equation (2.24), therefore,

$$c_4^{'} = 4h_{2i+1} + 4f_{i+1} + f_{i+1}^2 - f_i f_{i+2} + 1 + h_{2i+1}^2 - h_{2i}h_{2i+2} + 2f_{i+1}h_{2i+1} - f_{i+2}h_{2i} - 3h_{2i} - f_i h_{2i+2} - 3f_i - f_{i+2} - h_{2i+2}$$

$$= 4h_{2i+1} + f_{i+1} + 3f_{i-1} - f_{i+2} - (h_{2i}h_{2i+2} - h_{2i+1}^2) + 2f_{i+1}h_{2i+1} - f_{i+2}h_{2i} - 3h_{2i} - f_i h_{2i+2} - h_{2i+2}$$
$$= 2f_{i+1}h_{2i+1} + 4h_{2i+1} - f_{i+2}h_{2i} - 3h_{2i} - f_i h_{2i+2} - h_{2i+2} + 3f_{i-1} - f_i + (g_{n-1} + g_{n-3} + c_g) - 1 \quad (2.25)$$

$$w_1 = c_1 - c_4^{'} = 2f_i h_{2i+1} - f_{i+1}h_{2i} - f_{i-1}h_{2i+2} - (f_{i-1} + f_{i+1}) - 2(g_{n-1} + g_{n-3} + c_g) = h_{i+1} + h_{i-1} - l_{i-1} - 2(g_{n-1} + g_{n-3} + c_g) \quad (2.26)$$

Having given a detailed geometric proof for w_1 for Scenario I of the Theorem, it is assumed that the reader may be able to follow the same procedure in proving all eight scenarios.

It is in the interest of space management that we state Theorems 2.6 to 2.12 below without proof. The interested reader shall follow proof to Theorem 2.5.

Theorem 2.6

Let the sequence H_n be such that $h_i = g_{n+i-1} - f_i, i \geq 1,\ g_n = q_{m+n-1} + f_n, m, n \geq 4, n\ is\ odd.$ Consider the transformation $(h_i)_{i+1} \to (h_{i+1})_i$. Let the transformation vector be given by

$v = (w_1, w_2, w_3, w_4)$. When i is even,

$$\left.\begin{aligned} w_1 &= h_{i+1} + h_{i-1} - l_{i-1} - 2(g_{n-1} + g_{n-3} + c_g) \\ w_2 &= h_{i-2} + h_{i-4} - l_{i-4} + 2(g_{n-1} + g_{n-3} + c_g) \\ w_3 &= l_{i-1} - h_{i+1} - h_{i-1} - 2(g_{n-1} + g_{n-3} + c_g) \\ w_4 &= l_{i-4} - h_{i-2} - h_{i-4} + 2(g_{n-1} + g_{n-3} + c_g) \\ c_g &= g_n g_{n+2} - g_{n+1}^2 \end{aligned}\right\} \quad (2.27)$$

when i is odd,

$$\left.\begin{aligned} w_1 &= h_{i-2} + h_{i-4} - l_{i-4} - 2(g_{n-1} + g_{n-3} + c_g) \\ w_2 &= h_{i+1} + h_{i-1} - l_{i-1} + 2(g_{n-1} + g_{n-3} + c_g) \\ w_3 &= l_{i-4} - h_{i-2} - h_{i-4} - 2(g_{n-1} + g_{n-3} + c_g) \\ w_4 &= l_{i-1} - h_{i+1} - h_{i-1} + 2(g_{n-1} + g_{n-3} + c_g) \\ c_g &= g_n g_{n+2} - g_{n+1}^2 \end{aligned}\right\} \quad (2.28)$$

Theorem 2.7

Let the sequence H_n be such that $h_i = g_{n+i-1} - f_i, i \geq 1,\ g_n = q_{m+n-1} - f_n, m, n \geq 4, n\ is\ even.$ Consider the transformation $(h_i)_{i+1} \to (h_{i+1})_i$. Let the transformation vector be given by

$v = (w_1, w_2, w_3, w_4)$. When i is even,

$$\left.\begin{aligned} w_1 &= h_{i+1} + h_{i-1} - l_{i-1} - 2(g_{n-1} + g_{n-3} + c_g) \\ w_2 &= h_{i-2} + h_{i-4} - l_{i-4} + 2(g_{n-1} + g_{n-3} + c_g) \\ w_3 &= l_{i-1} - h_{i+1} - h_{i-1} - 2(g_{n-1} + g_{n-3} + c_g) \\ w_4 &= l_{i-4} - h_{i-2} - h_{i-4} + 2(g_{n-1} + g_{n-3} + c_g) \\ c_g &= g_n g_{n+2} - g_{n+1}^2 \end{aligned}\right\} \quad (2.29)$$

when i is odd,

$$\left.\begin{aligned} w_1 &= h_{i-2} + h_{i-4} - l_{i-4} - 2(g_{n-1} + g_{n-3} + c_g) \\ w_2 &= h_{i+1} + h_{i-1} - l_{i-1} + 2(g_{n-1} + g_{n-3} + c_g) \\ w_3 &= l_{i-4} - h_{i-2} - h_{i-4} - 2(g_{n-1} + g_{n-3} + c_g) \\ w_4 &= l_{i-1} - h_{i+1} - h_{i-1} + 2(g_{n-1} + g_{n-3} + c_g) \\ c_g &= g_n g_{n+2} - g_{n+1}^2 \end{aligned}\right\} \quad (2.30)$$

Theorem 2.8

Let the sequence H_n be such that $h_i = g_{n+i-1} - f_i, i \geq 1,\ g_n = q_{m+n-1} - f_n, m, n \geq 4, n\ is\ odd.$ Consider the transformation $(h_i)_{i+1} \to (h_{i+1})_i$. Let the transformation vector be given by

$v = (w_1, w_2, w_3, w_4)$. When i is even,

$$\left.\begin{aligned} w_1 &= h_{i+1} + h_{i-1} - l_{i-1} - 2(g_{n-1} + g_{n-3} + c_g) \\ w_2 &= h_{i-2} + h_{i-4} - l_{i-4} + 2(g_{n-1} + g_{n-3} + c_g) \\ w_3 &= l_{i-1} - h_{i+1} - h_{i-1} - 2(g_{n-1} + g_{n-3} + c_g) \\ w_4 &= l_{i-4} - h_{i-2} - h_{i-4} + 2(g_{n-1} + g_{n-3} + c_g) \\ c_g &= g_n g_{n+2} - g_{n+1}^2 \end{aligned}\right\} \quad (2.31)$$

when i is odd,

$$\left.\begin{aligned} w_1 &= h_{i-2} + h_{i-4} - l_{i-4} - 2(g_{n-1} + g_{n-3} + c_g) \\ w_2 &= h_{i+1} + h_{i-1} - l_{i-1} + 2(g_{n-1} + g_{n-3} + c_g) \\ w_3 &= l_{i-4} - h_{i-2} - h_{i-4} - 2(g_{n-1} + g_{n-3} + c_g) \\ w_4 &= l_{i-1} - h_{i+1} - h_{i-1} + 2(g_{n-1} + g_{n-3} + c_g) \\ c_g &= g_n g_{n+2} - g_{n+1}^2 \end{aligned}\right\} \quad (2.32)$$

Theorem 2.9

Let the sequence H_n be such that $h_i = g_{n+i-1} + f_i, i \geq 1,\ g_n = q_{m+n-1} + f_n, m, n \geq 4, n\ is\ even.$ Consider the transformation

$(h_i)_{i+1} \to (h_{i+1})_i$. *Let the transformation vector be given by*

$v = (w_1, w_2, w_3, w_4)$. *When i is even,*

$$\left.\begin{aligned} w_1 &= h_{i+1} + h_{i-1} - l_{i-1} + 2(g_{n-1} + g_{n-3} - c_g) \\ w_2 &= h_{i-2} + h_{i-4} - l_{i-4} - 2(g_{n-1} + g_{n-3} - c_g) \\ w_3 &= l_{i-1} - h_{i+1} - h_{i-1} + 2(g_{n-1} + g_{n-3} - c_g) \\ w_4 &= l_{i-4} - h_{i-2} - h_{i-4} - 2(g_{n-1} + g_{n-3} - c_g) \\ c_g &= g_n g_{n+2} - g_{n+1}^2 \end{aligned}\right\} \quad (2.33)$$

when i is odd,

$$\left.\begin{aligned} w_1 &= h_{i-2} + h_{i-4} - l_{i-4} + 2(g_{n-1} + g_{n-3} - c_g) \\ w_2 &= h_{i+1} + h_{i-1} - l_{i-1} - 2(g_{n-1} + g_{n-3} - c_g) \\ w_3 &= l_{i-4} - h_{i-2} - h_{i-4} + 2(g_{n-1} + g_{n-3} - c_g) \\ w_4 &= l_{i-1} - h_{i+1} - h_{i-1} - 2(g_{n-1} + g_{n-3} - c_g) \\ c_g &= g_n g_{n+2} - g_{n+1}^2 \end{aligned}\right\} \quad (2.34)$$

Theorem 2.10

Let the sequence H_n be such that $h_i = g_{n+i-1} + f_i, i \geq 1,\ g_n = q_{m+n-1} + f_n, m, n \geq 4, n\ is\ odd$. *Consider the transformation* $(h_i)_{i+1} \to (h_{i+1})_i$. *Let the transformation vector be given by*

$v = (w_1, w_2, w_3, w_4)$. *When i is even,*

$$\left.\begin{aligned} w_1 &= h_{i+1} + h_{i-1} - l_{i-1} + 2(g_{n-1} + g_{n-3} - c_g) \\ w_2 &= h_{i-2} + h_{i-4} - l_{i-4} - 2(g_{n-1} + g_{n-3} - c_g) \\ w_3 &= l_{i-1} - h_{i+1} - h_{i-1} + 2(g_{n-1} + g_{n-3} - c_g) \\ w_4 &= l_{i-4} - h_{i-2} - h_{i-4} - 2(g_{n-1} + g_{n-3} - c_g) \\ c_g &= g_n g_{n+2} - g_{n+1}^2 \end{aligned}\right\} \quad (2.35)$$

when i is odd,

$$\left.\begin{aligned} w_1 &= h_{i-2} + h_{i-4} - l_{i-4} + 2(g_{n-1} + g_{n-3} - c_g) \\ w_2 &= h_{i+1} + h_{i-1} - l_{i-1} - 2(g_{n-1} + g_{n-3} - c_g) \\ w_3 &= l_{i-4} - h_{i-2} - h_{i-4} + 2(g_{n-1} + g_{n-3} - c_g) \\ w_4 &= l_{i-1} - h_{i+1} - h_{i-1} - 2(g_{n-1} + g_{n-3} - c_g) \\ c_g &= g_n g_{n+2} - g_{n+1}^2 \end{aligned}\right\} \quad (2.36)$$

Theorem 2.11

Let the sequence H_n be such that $h_i = g_{n+i-1} + f_i, i \geq 1,\ g_n = q_{m+n-1} - f_n, m, n \geq 4, n\ is\ even$. *Consider the transformation* $(h_i)_{i+1} \to (h_{i+1})_i$. *Let the transformation vector be given by*

$v = (w_1, w_2, w_3, w_4)$. *When i is even,*

$$\left.\begin{aligned} w_1 &= h_{i+1} + h_{i-1} - l_{i-1} + 2(g_{n-1} + g_{n-3} - c_g) \\ w_2 &= h_{i-2} + h_{i-4} - l_{i-4} - 2(g_{n-1} + g_{n-3} - c_g) \\ w_3 &= l_{i-1} - h_{i+1} - h_{i-1} + 2(g_{n-1} + g_{n-3} - c_g) \\ w_4 &= l_{i-4} - h_{i-2} - h_{i-4} - 2(g_{n-1} + g_{n-3} - c_g) \\ c_g &= g_n g_{n+2} - g_{n+1}^2 \end{aligned}\right\} \quad (2.37)$$

when i is odd,

$$\left.\begin{aligned} w_1 &= h_{i-2} + h_{i-4} - l_{i-4} + 2(g_{n-1} + g_{n-3} - c_g) \\ w_2 &= h_{i+1} + h_{i-1} - l_{i-1} - 2(g_{n-1} + g_{n-3} - c_g) \\ w_3 &= l_{i-4} - h_{i-2} - h_{i-4} + 2(g_{n-1} + g_{n-3} - c_g) \\ w_4 &= l_{i-1} - h_{i+1} - h_{i-1} - 2(g_{n-1} + g_{n-3} - c_g) \\ c_g &= g_n g_{n+2} - g_{n+1}^2 \end{aligned}\right\} \quad (2.38)$$

Theorem 2.12

Let the sequence H_n be such that $h_i = g_{n+i-1} + f_i, i \geq 1,\ g_n = q_{m+n-1} - f_n, m, n \geq 4, n\ is\ odd$. *Consider the transformation* $(h_i)_{i+1} \to (h_{i+1})_i$. *Let the transformation vector be given by*

$v = (w_1, w_2, w_3, w_4)$. *When i is even,*

$$\left.\begin{aligned} w_1 &= h_{i+1} + h_{i-1} - l_{i-1} + 2(g_{n-1} + g_{n-3} - c_g) \\ w_2 &= h_{i-2} + h_{i-4} - l_{i-4} - 2(g_{n-1} + g_{n-3} - c_g) \\ w_3 &= l_{i-1} - h_{i+1} - h_{i-1} + 2(g_{n-1} + g_{n-3} - c_g) \\ w_4 &= l_{i-4} - h_{i-2} - h_{i-4} - 2(g_{n-1} + g_{n-3} - c_g) \\ c_g &= g_n g_{n+2} - g_{n+1}^2 \end{aligned}\right\} \quad (2.39)$$

when i is odd,

$$\left.\begin{aligned} w_1 &= h_{i-2} + h_{i-4} - l_{i-4} + 2(g_{n-1} + g_{n-3} - c_g) \\ w_2 &= h_{i+1} + h_{i-1} - l_{i-1} - 2(g_{n-1} + g_{n-3} - c_g) \\ w_3 &= l_{i-4} - h_{i-2} - h_{i-4} + 2(g_{n-1} + g_{n-3} - c_g) \\ w_4 &= l_{i-1} - h_{i+1} - h_{i-1} - 2(g_{n-1} + g_{n-3} - c_g) \\ c_g &= g_n g_{n+2} - g_{n+1}^2 \end{aligned}\right\} \quad (2.40)$$

2.3 Systems Evolution

A series of the transformations $(h_i)_{i+1} \to (h_{i+1})_i, i \geq 4$, for the sequence H_n yields very important results. For illustration we give the first ten such transformations for the sequence

$$H_n = 9,15,24,39,63,\ldots \quad (2.41a)$$

We only require the transformation vector, the reason for developing theorems 2.5 to 2.12. Because the sequence (2.41a) takes the form

$$h_i = g_{n+i-1} + f_i, i \geq 1, n = 5 \quad (2.41b)$$

where $G_n = F_n$, it follows that Theorem 2.10 and hence Equations (2.35) and (2.36) apply. These transformations are therefore:

$$\left.\begin{aligned} (h_4)_5 &\to (h_5)_4 \\ v &= (92,2,-60,-34) \end{aligned}\right\} \quad (2.42)$$

$$\left.\begin{array}{c}(h_5)_6 \to (h_6)_5 \\ v = (45,107,-13,-139)\end{array}\right\} \quad (2.43)$$

$$\left.\begin{array}{c}(h_6)_7 \to (h_7)_6 \\ v = (215,31,-183,-63)\end{array}\right\} \quad (2.44)$$

$$\left.\begin{array}{c}(h_7)_8 \to (h_8)_7 \\ v = (92,306,-60,-338)\end{array}\right\} \quad (2.45)$$

$$\left.\begin{array}{c}(h_8)_9 \to (h_9)_8 \\ v = (537,107,-505,-139)\end{array}\right\} \quad (2.46)$$

$$\left.\begin{array}{c}(h_9)_{10} \to (h_{10})_9 \\ v = (215,827,-183,-859)\end{array}\right\} \quad (2.47)$$

$$\left.\begin{array}{c}(h_{10})_{11} \to (h_{11})_{10} \\ v = (1380,306,-1348,-338)\end{array}\right\} \quad (2.48)$$

$$\left.\begin{array}{c}(h_{11})_{12} \to (h_{12})_{11} \\ v = (537,2191,-505,-2223)\end{array}\right\} \quad (2.49)$$

$$\left.\begin{array}{c}(h_{12})_{13} \to (h_{13})_{12} \\ v = (3587,827,-3555,-859)\end{array}\right\} \quad (2.50)$$

$$\left.\begin{array}{c}(h_{13})_{14} \to (h_{14})_{13} \\ v = (1380,5762,-1348,-5794)\end{array}\right\} \quad (2.51)$$

Recall $v = (w_1, w_2, w_3, w_4)$. w_1, w_3 in equation (2.45) equals w_1, w_3 in equation (2.42); w_2, w_4 in equation (2.46) equals w_2, w_4 in equation (2.43); w_1, w_3 in equation (2.47) equals w_1, w_3 in equation (2.44); w_2, w_4 in equation (2.48) equals w_2, w_4 in equation (2.45); etc. Theorems 2.13 and 2.14 structure this result.

Theorem 2.13

For the sequence H_n, let the transformation vector for the transformation

$(h_i)_{i+1} \to (h_{i+1})_i, i \geq 4,$ *be given by* $v = (w_1, w_2, w_3, w_4)$ *. If i is even, then* $(h_{i+3})_{i+4} \to (h_{i+4})_{i+3}$ *has vector* $v' = (w_1', w_2', w_3', w_4')$ *such that* $w_1' = w_1;\ w_3' = w_3.$

Proof

Assume theorem 2.9 applies. In the transformation $(h_i)_{i+1} \to (h_{i+1})_i$, if i is even, equation (2.33) gives

$$\left.\begin{array}{l}w_1 = h_{i+1} + h_{i-1} - l_{i-1} + 2(g_{n-1} + g_{n-3} - c_g) \\ w_3 = l_{i-1} - h_{i+1} - h_{i-1} + 2(g_{n-1} + g_{n-3} - c_g)\end{array}\right\} \quad (2.52)$$

Since i is even, it follows $(i+3)$ is odd. Let $j = (i+3)$. We need the transformation $(h_j)_{j+1} \to (h_{j+1})_j$. Since j is odd, from equation (2.34),

$$\left.\begin{array}{l}w_1' = h_{j-2} + h_{j-4} - l_{j-4} + 2(g_{n-1} + g_{n-3} - c_g) \\ w_3' = l_{j-4} - h_{j-2} - h_{j-4} + 2(g_{n-1} + g_{n-3} - c_g)\end{array}\right\} \quad (2.53)$$

But $j = (i+3)$, it follows

$$\left.\begin{array}{l}w_1' = h_{i+1} + h_{i-1} - l_{i-1} + 2(g_{n-1} + g_{n-3} - c_g) \\ w_3' = l_{i-1} - h_{i+1} - h_{i-1} + 2(g_{n-1} + g_{n-3} - c_g)\end{array}\right\} \quad (2.54)$$

Notice that Equations (2.52) and (2.54) are equal, therefore result is true.

Remark

In our proof we have assumed that theorem 2.9 applies. The same can be done with any of theorems 2.5 to 2.12.

Theorem 2.14

For the sequence H_n, let the transformation vector for the transformation $(h_i)_{i+1} \to (h_{i+1})_i, i \geq 4$, be given by $v = (w_1, w_2, w_3, w_4)$. If i is even, then $(h_{i+3})_{i+4} \to (h_{i+4})_{i+3}$ has vector $v' = (w_1', w_2', w_3', w_4')$ such that $w_1' = w_1;\ w_3' = w_3$.

Proof

Assume Theorem 2.12 applies. In the transformation $(h_i)_{i+1} \to (h_{i+1})_i$, when i is odd, equation (2.40) gives

$$\left.\begin{array}{l}w_2 = h_{i+1} + h_{i-1} - l_{i-1} - 2(g_{n-1} + g_{n-3} - c_g) \\ w_4 = l_{i-1} - h_{i+1} - h_{i-1} - 2(g_{n-1} + g_{n-3} - c_g)\end{array}\right\} \quad (2.55)$$

Since i is odd, it follows $(i+3)$ is even. Let $j = (i+3)$. For the transformation $(h_j)_{j+1} \to (h_{j+1})_j$, since j is even, from equation (2.39),

$$\left.\begin{array}{l}w_2' = h_{j-2} + h_{j-4} - l_{j-4} - 2(g_{n-1} + g_{n-3} - c_g) \\ w_4' = l_{j-4} - h_{j-2} - h_{j-4} - 2(g_{n-1} + g_{n-3} - c_g)\end{array}\right\} \quad (2.56)$$

With $j = i + 3$, we have

$$\left.\begin{array}{l}w_2' = h_{i+1} + h_{i-1} - l_{i-1} - 2(g_{n-1} + g_{n-3} - c_g) \\ w_4' = l_{i-1} - h_{i+1} - h_{i-1} - 2(g_{n-1} + g_{n-3} - c_g)\end{array}\right\} \quad (2.57)$$

Notice that (2.57)=(2.55), therefore result is true.

Here we have a system that retains and modifies certain attributes as it evolves. But equally striking is the fact that this concept provides once again a link between the golden section and the Teleois numerical system:

$$\left.\begin{array}{r} T_n = 1,4,7,10,13,\ldots \\ t_{n+2} = t_{n+1} + 3, n \geq 1, t_1 = 1 \end{array}\right\} \quad (2.58)$$

As implied by theorems 2.13 and 2.14, replication and modification of attributes occurs at Teleois positions. Therefore by coding a system following the concept of this paper one does not only implement the golden section, but the Teleois also, about which Hardy et al. [13], cited by Sherbon [14] say, *"Understand the electromagnetic frequencies of the atom and you understand why the Teleois proportions were used."*

3. CONCLUSION

Our theorems demonstrate that the golden section can be implemented in the design of evolving systems, which concept should be appreciated by both geometer and computer expert, with applications range from information technology to manufacturing. Albeit the emphasis is on applications, the results may be used for further development of the theory of the golden section.

COMPETING INTERESTS

Author has declared that no competing interests exist.

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